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STUDIES IN GRAVITATIONAL COLLAPSE

by

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies & Research for acceptance, a thesis entitled "STUDIES IN GRAVITATIONAL COLLAPSE" submitted by JAMES E. CHASE in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

The relativistic gravitational collapse of an object is usually considered in four stages: instability, implosion, horizon, and singularity. In addition, the collapse itself falls into one of three categories: spherical, nearly-spherical, or highly nonspherical. This thesis is devoted to the presentation of three studies, each of which deals with a particular phase of this complex problem.

In the first study we present some fairly elementary results, related to the instability stage of spherical collapse. For simplicity we deal with spherical shells. A solution of the Einstein equations is derived, representing a thin spherical shell of charged fluid, falling in a spherically symmetric field due to mass and charge at its centre. No restrictions are placed on the equation of state. We integrate the equations of motion to find the law of conservation of total energy, and we use it to study the equilibrium states of the system, and their stability against collapse. We find under reasonable assumptions, that, given the entropy and the equation of state, there is a maximum equilibrium mass, and, corresponding to it, a critical radius, inside of which instability sets in. For uncharged bodies, these results completely parallel, and serve as a simple illustration of, the much more complicated analyses needed for fluid spheres.

The second study is concerned with the implosion and horizon stages of nearly-spherical collapse. Two idealized collapse models (again

thin shells), involving a scalar monopole and a magnetic dipole, are considered, treating departures from sphericity as small perturbations. Radiative leakage (largely downwards through the Schwarzschild horizon) causes the externally observable asymmetries to decay to zero in an oscillatory fashion. These results have significant consequences for astrophysics; they imply in particular that a "black hole" cannot be a source of synchrotron radiation.

In the third study we deal mainly with the horizon stage of highly nonspherical collapse, and we consider static scalar fields. We prove the following theorem. Every zero-mass scalar field which is gravitationally coupled, static and asymptotically flat, becomes singular at a simply-connected event horizon. In the special case where the gravitational coupling of the scalar energy density is neglected, the solutions are computed explicitely. Some properties of the singular event horizons are also discussed.

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CHAPTER I

Introduction

§1.1 Spherical Collapse

The story of relativistic gravitational collapse really began in 1916, when the normally formidible Einstein equations were shown, by Schwarzschild [1], to yield, in the static spherically symmetric vacuum case, a beautifully simple solution,

$$ds^2 = -(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \quad . \qquad (1.1)$$

(We choose the signature to be $- + + +$ and our units such that $G = c = 1$.)

Several of the well-known properties of the metric form (1.1) are worth noting.

(i) Recall that (1.1) is static. It can be shown that "any spherically symmetric vacuum field is static". This result is known as Birkhoff's Theorem [2], and in particular it tells us that, even in the case of a star undergoing catastrophic spherical collapse, the external line element is just the Schwarzschild metric (1.1).

(ii) The apparent singularity at $r = 2m$ is <u>not</u> a physical singularity, but merely a pathology of the particular coordinate system chosen - in fact, by transforming to different

coordinates, or examining scalars formed from the curvature tensor, it is easily seen that the space-time is perfectly well-behaved at $r = 2m$ [3]. On the other hand, the $r = 2m$ hypersurface does have special properties. It is an "event horizon" [4], i.e. it is the inner boundary of the set of events which can be connected to spatial infinity by future timelike lines. Also, every spacelike 2-sphere S^2 ($r = \text{const.} < 2m$, $t = \text{const.}$) inside the $r = 2m$ hypersurface is "trapped", i.e. both systems of null geodesics (outgoing as well as incoming), which meet S^2 orthogonally, converge locally in future directions [5].

(iii) The singularity at $r = 0$ is a true physical one [6], i.e. a free-falling observer would be torn apart by tidal gravitational forces near $r = 0$.

Since the gravitational surface $r = 2m$ is not physically singular, there is no physical reason why a star should not fall inside it. At the same time, once a star is inside its own gravitational radius, it can never reemerge (property (ii)), hence collapse is irreversible. In this sense then, general relativity predicts the inevitability of spherical stellar collapse, and these predictions have been verified, for certain idealized models, by several calculations [7].

The overall phenomenon of spherical gravitational collapse has been portrayed by Thorne [8] as having four key stages:

(i) Instability. At a certain stage in its evolution, the star runs out of nuclear fuel, which affects the equation of state and thus leads to instability.

(ii) Implosion. In a very short time the instability causes collapse to begin, with the dense core falling more rapidly and leaving an outer envelope to trail behind.

(iii) Horizon. The stellar surface reaches its gravitational radius in a finite time, as measured by a comoving observer. However, to a distant observer, the star takes infinite time to approach $r = 2m$, i.e. this surface is, for him, an event horizon. This horizon becomes the boundary of a "black hole", a region of space-time that can never communicate with the "outside world".

(iv) Singularity. The star continues to collapse and soon reaches the singularity at $r = 0$, corresponding to infinite density and infinite tidal gravitational forces.

§1.2 Nearly-Spherical Collapse.

A fundamental problem still remains. Does our present description of idealized spherical collapse also fit, to any degree of accuracy, the case of a more realistic, non-spherically collapsing star? In other words, will a star, whose collapse is nonspherical because of rotation, magnetic fields, asymmetric density, etc., behave qualitatively like the

idealized spherical case? Or is it possible that these complications might alter the spherical collapse picture by stopping or reversing the collapse [9] or by eliminating the event horizon [10], thus leaving the $r = 0$ singularity visible to all?

A complete solution to this problem still is not known, although several recent results have provided us with a good picture of collapse, in the special case of small perturbations from sphericity. We will briefly outline these developments now.

In 1963, we were again very fortunate when the Einstein equations miraculously yielded another simple solution, namely, that of Kerr [11], for stationary axi-symmetric vacuum fields. This was generalized to include the case of charge, a short time later [12].

Early studies of perturbed spherical collapse, while lacking detail, indicated that: (i) the external field of a collapsing star should become asymptotically stationary [13];

(ii) if we assume (i), then in order to avoid singularities [14], all externally-viewed asymmetries must somehow disappear [15] as the collapsing object approaches the horizon.

More recently, these results have been largely confirmed. Novikov [16] proves that, in fact, the perturbations do not become singular near the horizon. De la Cruz, Chase, and Israel [17] numerically trace the perturbed collapse of several simple idealized models, and show that, as predicted, the perturbations do not blow up, but are "radiated away" as gravitational, electromagnetic, or scalar waves - some going off to

infinity, and the rest falling in through the horizon.

The definitive contribution in this area is undoubtedly the beautiful work of Price [18]. He also carries out numerical analyses of perturbed collapse, and is the first to succeed in showing how the "tails" of the radiated waves die out, for large t. Even more important, he proves a theorem that explains exactly why the perturbations are radiated away.

We can therefore summarize nonspherical gravitational collapse, in the case of small perturbations, in the following way:

(i) Nearly-sherical collapse <u>does</u> behave qualitatively like idealized spherical collapse (except possibly for the "singularity" stage, about which little is known). In other words, perturbed spherical collapse also fits into the standard mold - instability, implosion, horizon.

(ii) Price's Theorem: "In relativistic gravitational collapse with small nonspherical deformations, anything that can be radiated, will be radiated away completely." This theorem holds for zero-rest-mass fields of arbitrary integer spin. It is well known that all multipole moments of order $\ell \geq s$ (for a massless spin-s field) can be radiated away - the rest cannot, i.e. they are "conserved". According to Price's theorem, then, the end result of nearly-spherical collapse must be a "black hole" exhibiting <u>no</u> scalar field, only a monopole electromagnetic field (charge), only monopole and

dipole gravitational fields [19], and so on.

§1.3 Highly Nonspherical Collapse [20]

The question of whether or not black holes still are formed, when the collapse is of a highly nonspherical nature, has become a widely investigated problem in recent times - a problem not yet fully resolved. Nevertheless, mounting evidence, while still not conclusive, points to the following results:

(i) "The Black Hole Conjecture". A black hole forms when and only when a mass m is somehow compressed until its circumference in <u>every</u> direction is $C \lesssim 4\pi m$.

(ii) "A black hole has no hair". This colourful expression of J.A. Wheeler memorably paraphrases the conjecture that, after collapse, the horizon will eventually settle down to a stationary state, whose external field is given by the "charged Kerr" metric. In other words, a black hole, and hence its exterior geometry, is ultimately determined by just three "conserved quantities" - the mass m , charge e , and angular momentum $\underset{\sim}{J}$ that fall into the hole. Other factors, such as the collapsing body's asymmetries (e.g. gravitational quadrupole), or locally conserved quantities (e.g. baryon number), have no net effect on the external field [21].

We will briefly review the theorems and analyses that tend to support these conjectures. The main evidence in support of (i) is naturally its well-established truth for spherical and almost-spherical collapse. Several examples of highly nonspherical collapse have been demonstrated, in which no horizon is formed [22]. However, these examples are pathological in nature, and the absence of a horizon seems due to the fact that the collapse does not occur in all directions (hence the inclusion of the phrase "in every direction" in the black hole conjecture).

The plausibility of conjecture (ii) is enhanced by a series of rigourously established results, concerning static and stationary event horizons. We can summarize the static $(\underset{\sim}{J} = 0)$ results as follows:

(i) The Schwarzschild fields with $m \geq 0$ are the only static, vacuum $(e = 0)$ solutions which are asymptotically flat and possess simply-connected equipotential 2-surfaces $(g_{oo} = \text{const.})$, and a nonsingular horizon $g_{oo} = 0$ [23].

(ii) The Reissner-Nordstrom solutions with $m \geq |e|$ occupy the same unique position in the class of all static electrovac solutions [24].

(iii) Every zero-mass scalar field which is gravitationally coupled, static and asymptotically flat, becomes singular at a simply-connected event horizon [25].

From these theorems we would like to conclude that "a nonrotating black hole has no hair", i.e. it is completely determined by the mass m and charge e (if any) that goes down the hole. Unfortunately, we cannot draw this conclusion, for two reasons. First, the topological requirement that the equipotential 2-surfaces be simply-connected, is overly restrictive, and should be dropped. Second, the demand for static solutions is too strong and must be replaced by the weaker conditions "stationary plus $\underset{\sim}{J} = 0$ ". A definite improvement, in the uncharged case, has been made by Carter [26] who reproves result (i) with the changes indicated, except that he needs to assume axial symmetry - an unfortunate restriction which no one has yet succeeded in removing.

In the case where angular momentum is nonvanishing, established results have been found only recently. Carter [27] was the first to provide valuable evidence in support of our conjecture. He again considers the charge-free case, and shows that all asymptotically flat, stationary vacuum fields, satisfying several additional (and unwanted) requirements including axial symmetry, comprise a 2-parameter family, where one of the parameters is the magnitude of the angular momentum. Moreover, only the family of Kerr solutions contains a member with $\underset{\sim}{J} = 0$, i.e. only the Kerr family contains the Schwarzschild subfamily.

Very recently Hawking [28] has proved the following important result: every stationary field containing an event horizon, which consists of past and future sheets intersecting in a 2-space with the topology of a 2-sphere, and which is vacuum in some neighbourhood of the horizon, is either static or axisymmetric. This theorem can now be combined with the earlier

theorems of Israel (static) or Carter (axisymmetric) to "almost prove" (we still have several unwanted requirements) that the field must be Schwarzschild or Kerr, respectively.

If this "combined theorem" eliminating the possibility of other 2-parameter families distinct from the Kerr family, can be reproved, without the unwanted conditions, it will establish that a stationary vacuum black hole is a Kerr black hole, and from there, it seems very reasonable that the external field of a stationary electrovac black hole will be of the "charged Kerr" type. However, at present these remain unverified conjectures.

§1.4 Summary of the Thesis.

The previous three sections have provided us with a brief survey of the overall field of relativistic gravitational collapse. The purpose of this dissertation is to present several separate results - each of which deals with a particular phase of this complex problem.

In Chapter II we present some fairly elementary results, related to the instability stage of spherical collapse. For simplicity we deal with spherical shells. A solution of the Einstein equations is derived, representing a thin spherical shell of charged fluid, falling in a spherically symmetric field due to mass and charge at its centre. No restrictions are placed on the equation of state. We integrate the equations of motion to find the law of conservation of total energy, and we use it to study the equilibrium states of the system, and their stability

against collapse. We find, under reasonable assumptions, that, given the entropy and the equation of state, there is a maximum equilibrium mass, and, corresponding to it, a critical radius, inside of which instability sets in. For uncharged bodies, these results completely parallel, and serve as a simple illustration of, the much more complicated analyses needed for fluid spheres.

Chapter III contains studies concerned with the implosion and horizon stages of nearly-spherical collapse. Two idealized collapse models (again thin shells), involving a scalar monopole and a magnetic dipole, are considered, treating departures from sphericity as small perturbations. Radiative leakage (largely downwards through the Schwarzschild horizon) causes the externally observable asymmetries to decay to zero in an oscillatory fashion. These results have significant consequences for astrophysics; they imply in particular that a black hole cannot be a source of synchrotron radiation.

In Chapter IV we deal mainly with the horizon stage of highly nonspherical collapse, in the case of static scalar fields. We prove the following theorem. Every zero-mass scalar field which is gravitationally coupled, static and asymptotically flat, becomes singular at a simply-connected event horizon. In the special case where the gravitational coupling of the scalar energy density is neglected, the solutions are computed explicitely. Some properties of the singular event horizons are also discussed.

CHAPTER II

Gravitational Instability and Collapse of Charged Fluid Shells

§2.1 Introduction.

In this chapter we consider the motion of a spherical shell of ideal fluid, with charge e and mass m, falling in the external field due to a spherical distribution of gravitational mass m_1 and charge e_1 at its centre. In Newtonian theory, the dynamics of this system are described by

$$T + U + (e_2-e_1)^2/2R + (e_2-e_1)e_1/R - (m_2-m_1)^2/2R$$
$$- (m_2-m_1)m_1/R = \text{const.} \ ,$$

expressing conservation of total energy, along with the adiabatic condition

$$dU = -P\ d(4\pi R^2) \quad ,$$

where T = kinetic energy , U = internal energy, P = surface pressure, $e_2 = e+e_1$, $m_2 = m+m_1$.

It is our aim to establish the relativistic analogue of this, namely

$$\left.\begin{aligned}
&1 + \dot{R}^2 = A + B/R + C/R^2 \quad , \\
&A = (m_2-m_1)^2/M^2 \ , \ B = m_1+m_2 - (m_2-m_1)(e_2{}^2-e_1{}^2)/M^2 \\
&C = (e_2{}^2-e_1{}^2)^2/4M^2 + M^2/4 - (e_1{}^2+e_2{}^2)/2 \quad ,
\end{aligned}\right\} \tag{2.1}$$

where A , B , C are functions of $M = M_o + U$, and $dM = -Pd(4\pi R^2)$. In what follows, we will associate with the shell three different "masses": M_o (sum of the rest masses of the constituent particles of the shell), the total proper mass M (the "bare or baryon mass" M_o plus the internal thermal energy of the shell), and the gravitational mass m (the total energy of the shell). M_o and m are then constants of the motion.

Although the problem of gravitational collapse of a thin shell is a hypothetical one, from a physical point of view, it is nevertheless useful, because in its crudest overall aspects, the dynamics of stellar collapse turns out to be not too sensitive to the detailed distribution of matter within the star. The advantage of such a shell model is that a complete solution can be easily found in an explicit form [29].

Various special cases of the general result (2.1) have been obtained previously. Israel [30] has derived the equation of motion of a spherical shell of incoherent dust ($m_1=e_1=e_2=0$; zero pressure) and studied its collapse [31]. Using a similar approach, de la Cruz and Israel [32] have dealt with a charged shell of dust in the field due to

a concentration of mass and charge near its centre (zero pressure). Papapetrou and Hamoui [33] have studied uncharged shells of ideal fluid $(m_1=e_1=e_2=0)$ and their motions, assuming a particular equation of state [34]. Kuchar [35] has considered charged fluid shells $(m_1=e_1=0)$, but restricts the equation of state to be polytropic. We propose to obtain (2.1) without placing any restriction on the equation of state.

After some preliminaries (§2.2), the equations of motion for the system are derived (§2.3) and integrated (§2.4), giving (2.1). In §2.5, we deal with the stability of an uncharged fluid shell $(m_1=e_1=e_2=0)$ in equilibrium. A brief treatment of stability for the more complicated original system is given in §2.6. At several points we refer to Appendix A, which contains some results of two-dimensional relativistic gas theory, necessary in this work.

§2.2 Geometrical and Dynamical Preliminaries.

A brief sketch of the geometry and dynamics of a thin shell of ideal fluid is offered here [36].

Let Σ be a timelike hypersurface, dividing space-time into two 4-dimensional domains, V_+ and V_-, and belonging to the boundary of both [37]. Suppose ξ^i are intrinsic coordinates for Σ, with $\underset{\sim}{e}_{(i)}$ the associated tangent base vectors, and $\underset{\sim}{n}$ a unit spacelike normal to Σ (directed from V_- to V_+). For an arbitrary vector field $\underset{\sim}{A}$, the intrinsic covariant derivative with respect to ξ^j is defined by

$$A_{i;j} = \underset{\sim}{e}_{(i)} \cdot \partial \underset{\sim}{A}/\partial \xi^j = \partial A_i/\partial \xi^j - A^k \Gamma_{k,ij} \quad , \tag{2.2}$$

where the Christoffel symbols $\Gamma_{k,ij}$ are given by

$$\Gamma_{k,ij} = \underset{\sim}{e}_{(k)} \cdot \partial \underset{\sim}{e}_{(i)}/\partial \xi^j \quad . \tag{2.3}$$

Non-intrinsic properties enter by means of the extrinsic curvature 3-tensor of Σ, which measures the variations $\partial \underset{\sim}{n}/\partial \xi^i$ of the unit normal. Each of these three vectors is perpendicular to $\underset{\sim}{n}$ (hence tangent to Σ), so we may write

$$\partial \underset{\sim}{n}/\partial \xi^i = K_i^{\ j} \underset{\sim}{e}_{(j)} \quad , \tag{2.4}$$

which leads to

$$K_{ij} = \underset{\sim}{e}_{(j)} \cdot \partial \underset{\sim}{n}/\partial \xi^i = -\underset{\sim}{n} \cdot \partial \underset{\sim}{e}_{(j)}/\partial \xi^i = -\underset{\sim}{n} \cdot \partial \underset{\sim}{e}_{(i)}/\partial \xi^j = K_{ji} \quad . \tag{2.5}$$

In general, when measured with respect to V_+ and V_-, the corresponding components of this tensor, K^+_{ij} and K^-_{ij}, may or may not be equal. However, in order that Σ be (the history of) a surface layer [38], we require $K^+_{ij} \neq K^-_{ij}$. It then follows, provided

$$\gamma_{ij} \equiv K^+_{ij} - K^-_{ij} = [K_{ij}] \tag{2.6}$$

is nonvanishing, that Σ is the history of a thin shell.

From (2.3) and (2.5) we obtain the Gauss-Weingarten equations

$$\partial \underset{\sim}{e}_{(i)}/\partial \xi^j = - K_{ij} \underset{\sim}{n} + \Gamma^k_{ij} \underset{\sim}{e}_{(k)} \quad , \tag{2.7}$$

which, with (2.2) leads to

$$\partial \underset{\sim}{A}/\partial \xi^j = A^i{}_{;j} \underset{\sim}{e}_{(i)} - A^i K_{ij} \underset{\sim}{n} \quad . \tag{2.8}$$

Taking $\partial/\partial\xi^1$ of (2.7), using (2.4) and (2.7), and applying the Ricci commutation relations [39] we obtain the well-known Gauss-Codazzi equations

$$R_{\alpha\beta\gamma\delta}\, e^\alpha_{(a)} e^\beta_{(b)} e^\gamma_{(c)} e^\delta_{(d)} = R_{abcd} - K_{ac}K_{bd} + K_{bc}K_{ad} \quad , \tag{2.9}$$

$$R_{\alpha\beta\gamma\delta}\, n^\alpha\, e^\beta_{(b)} e^\gamma_{(c)} e^\delta_{(d)} = K_{bc;d} - K_{bd;c} \quad . \tag{2.10}$$

Operating on (2.9) and (2.10) with $g^{bc}g^{ad}$ and g^{bd} respectively, and using the fact that

$$g^{bc} e^\beta_{(b)} e^\gamma_{(c)} = g^{\beta\gamma} - n^\beta n^\gamma \quad , \tag{2.11}$$

we find

$$\left. {}^3R - K_{ab}K^{ab} + K^2 \right|^{+}_{-} = - 2\, G_{\alpha\beta}\, n^\alpha n^\beta \Big|^{+}_{-} \quad , \tag{2.12}$$

$$\left. K_a{}^b{}_{;b} - K_{;a} \right|^{+}_{-} = - G_{\alpha\beta}\, e^\alpha_{(a)} n^\beta \Big|^{+}_{-} \quad , \tag{2.13}$$

where ${}^3R$ is the intrinsic 3-curvature invariant of $\sum$, $K = g^{ab}K_{ab}$, and $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}\, g_{\alpha\beta}\, R$ is the Einstein tensor.

Let $\underset{\sim}{u}$ be a unit vector, tangent to Σ, representing the velocity of matter in the shell. Then, from (2.8), the 4-acceleration of an element of the shell, as measured in V_+ and V_-, is [40]

$$\delta u^\alpha/\delta\tau\Big|^{\pm} = e^\alpha_{(i)}\; u^i{}_{;j}u^j - n^\alpha K_{ij}u^iu^j\Big|^{\pm} \quad . \tag{2.14}$$

Since any motions of the shell are in a direction normal to the shell, we restrict our interest to the normal components of (2.14), namely

$$n_\alpha \delta u^\alpha/\delta\tau\Big|^{\pm} = -u^iu^j\overset{\pm}{K}_{ij} \quad ,$$

from which, by adding and subtracting, we obtain

$$n_\alpha \delta u^\alpha/\delta\tau\Big|^{+} + n_\alpha \delta u^\alpha/\delta\tau\Big|^{-} = -\,2u^iu^j\tilde{K}_{ij} \quad , \tag{2.15}$$

$$n_\alpha \delta u^\alpha/\delta\tau\Big|^{+} - n_\alpha \delta u^\alpha/\delta\tau\Big|^{-} = -\,u^iu^j\;\gamma_{ij} \quad \text{(from (2.6))}. \tag{2.16}$$

Turning to dynamical considerations, the components of the surface energy 3-tensor are defined by the "Lanczos equations",

$$\gamma_{ij} - g_{ij}\gamma = -8\pi S_{ij} \quad , \tag{2.17}$$

or equivalently

$$\gamma_{ij} = -8\pi(S_{ij} - \frac{1}{2}\, g_{ij}\, S) \quad . \tag{2.18}$$

We also have Einstein's field equations,

$$G_{\alpha\beta} = -8\pi T_{\alpha\beta} \quad , \tag{2.19}$$

satisfied in the regions V_- and V_+, exterior to the shell. Using (2.17) and (2.19), the jumps of (2.12) and (2.13) across $\sum$ can be put in the form

$$S^{ij}\tilde{K}_{ij} = [T_{\alpha\beta}n^\alpha n^\beta] \quad , \tag{2.20}$$

$$S^j_{i;j} = -[T_{\alpha\beta}e^\alpha_{(i)}n^\beta] \quad . \tag{2.21}$$

For a shell of ideal fluid, the intrinsic surface energy tensor has the form

$$S^{ij} = (\sigma+P)\,u^i u^j + Pg^{ij} \quad , \quad u^i u_i = -1 \quad , \tag{2.22}$$

where P is the surface pressure, and σ is the surface energy density of the fluid. It is easily shown from (2.20) and (2.22) that

$$-2u^i u^j \tilde{K}_{ij} = 2(\sigma+P)^{-1}\,\{P\tilde{K}-[T_{\alpha\beta}n^\alpha n^\beta]\} \quad ,$$

and from (2.18) and (2.22) that

$$-u^i u^j \gamma_{ij} = 8\pi(P+\sigma/2) \quad .$$

Hence (2.15) and (2.16) can be written

$$n_\alpha \delta u^\alpha/\delta\tau\Big|^{+} + n_\alpha \delta u^\alpha/\delta\tau\Big|^{-} = 2(\sigma+P)^{-1}\,\{P\tilde{K}-[T_{\alpha\beta}n^\alpha n^\beta]\} \quad , \tag{2.23}$$

$$n_\alpha \delta u^\alpha/\delta\tau\Big|^{+} - n_\alpha \delta u^\alpha/\delta\tau\Big|^{-} = 8\pi(P+\sigma/2) \quad . \tag{2.24}$$

Note that these results agree with those of de la Cruz and Israel [41] for zero pressure, and are identical with Kuchar's results [42].

For future reference, it is easily derived from (2.21) and (2.22) that

$$(\sigma u^j)_{;j} + P u^j_{;j} = [T_{\alpha\beta} e^\alpha_{(i)} n^\beta]\; u^i = [T_{\alpha\beta} u^\alpha n^\beta] \quad . \tag{2.25}$$

§2.3 Charged Spherical Shell of Ideal Fluid in a Spherisymmetric Electrovac Field.

Consider a charged shell of ideal fluid, falling in the field due to a spherically symmetric distribution of mass m_1 and charge e_1, near its centre. Let $r = R(\tau)$ be the equation of $\sum$. Then its intrinsic metric is given by

$$(ds^2)_\sum = \{R(\tau)\}^2\, d\Omega^2 - d\tau^2 \qquad (d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2) \quad , \tag{2.26}$$

where $\xi^i = (\theta,\phi,\tau)$ and τ is proper time along the streamlines θ,ϕ = constant. In both V_+ and V_-, the line element is reducible to the Reissner-Nordstrom metric, by extension of Birkhoff's theorem [43]. Hence we have

$$(ds^2)_- = \{f_-(r)\}^{-1} dr^2 + r^2 d\Omega^2 - f_-(r) dt_-^2 \qquad (r < R(\tau)) \quad , \tag{2.27}$$

$$(ds^2)_+ = \{f_+(r)\}^{-1} dr^2 + r^2 d\Omega^2 - f_+(r) dt_+^2 \qquad (r > R(\tau)) \quad , \tag{2.28}$$

where

$$f_-(r) = 1-2m_1/r + e_1^2/r^2 \quad , \quad f_+(r) = 1-2m_2/r + e_2^2/r^2 \ . \qquad (2.29)$$

It is easily seen, by proper choice of $t_-(\tau)$, $t_+(\tau)$ on $\sum$, that the metrics induced by (2.27) and (2.28) on $\sum$, agree with (2.26).

It can be verified, by intrinsically differentiating $u^\alpha u_\alpha = -1$, with $\dot{R} = dR/d\tau$ and $\dot{t}_+ = dt_+/d\tau$, that [44]

$$0 = u_\alpha \delta u^\alpha/\delta\tau\Big|^+ = f_+(R)^{-1}\, \dot{R}\ \delta\, \dot{R}/\delta\tau - \{f_+(R) + \dot{R}^2\}^{\frac{1}{2}}\, \delta\, \dot{t}_+/\delta\tau \ . \qquad (2.30)$$

We now use (2.30) to eliminate $\delta\dot{t}_+/\delta\tau$ from

$$n_\alpha \delta u^\alpha/\delta\tau\Big|^+ = \{f_+(R) + \dot{R}^2\}^{\frac{1}{2}}\, f_+(R)^{-1}\, \delta\, \dot{R}/\delta\tau - \dot{R}\, \delta\, \dot{t}_+/\delta\tau \quad ,$$

and since

$$\delta\dot{R}/\delta\tau = \ddot{R} + \frac{1}{2}\, df_+(R)/dR \quad ,$$

we finally obtain, from (2.29),

$$n_\alpha \delta u^\alpha/\delta\tau\Big|^+ = \{f_+(R) + \dot{R}^2\}^{-\frac{1}{2}}\, \{\ddot{R} + m_2/R^2 - e_2^2/R^3\} \ . \qquad (2.31)$$

The analogous expression for $n_\alpha \delta u^\alpha/\delta\tau\Big|^-$ is

$$n_\alpha \delta u^\alpha/\delta\tau\Big|^- = \{f_-(R) + \dot{R}^2\}^{-\frac{1}{2}}\, \{\ddot{R} + m_1/R^2 - e_1^2/R^3\} \ . \qquad (2.32)$$

The energy tensor associated with the Reissner-Nordstrom metric (2.27) is given by

$$-T^4_4 = -T^1_1 = T^2_2 = T^3_3 = e_1^2/8\pi r^4 \quad \text{(other components zero)}, \tag{2.33}$$

with e_1 replaced by e_2 for the metric (2.28). From (2.33) it is easily shown that

$$T^\beta_\alpha u^\alpha n_\beta \Big|^{+}_{-} = 0 \quad , \tag{2.34}$$

$$[T_{\alpha\beta} n^\alpha n^\beta] = -(e_2^2 - e_1^2)/8\pi R^4 \quad . \tag{2.35}$$

Using (2.25) and (2.34) it follows that

$$(\sigma u^j)_{;j} = -P u^j_{;j} \quad . \tag{2.36}$$

The left side of (2.36) is just the rate of increase of surface energy, while the right side is equal to minus the rate of work done by the pressure in expanding the shell. Hence, unlike the de la Cruz and Israel result for dust [45], the proper mass of an element of fluid under surface pressure P changes with time.

We can now write (2.23) and (2.24), using (2.31), (2.32) and (2.35), in the form

$$\{f_+(R) + \dot{R}^2\}^{-\frac{1}{2}} \{\ddot{R} + m_2/R^2 - e_2^2/R^3\} + \{f_-(R) + \dot{R}^2\}^{-\frac{1}{2}} \{\ddot{R} + m_1/R^2 - e_1^2/R^3\}$$

$$= 2(\sigma+P)^{-1}\{P\tilde{K} + (e_2^2 - e_1^2)/8\pi R^4\} \ ,$$

(2.37)

$$\{f_+(R) + \dot{R}^2\}^{-\frac{1}{2}} \{\ddot{R}+m_2/R^2 - e_2^2/R^3\} - \{f_-(R) + \dot{R}^2\}^{-\frac{1}{2}} \{\ddot{R}+m_1/R^2 - e_1^2/R^3\}$$

$$= 8\pi(P+\sigma/2) \quad . \tag{2.38}$$

We call (2.37) and (2.38) the equations of motion for the system. In order to simplify the notation, we define $F_+ = \{f_+(R) + \dot{R}^2\}^{1/2}$ and $F_- = \{f_-(R) + \dot{R}^2\}^{1/2}$. Then (2.37) and (2.38) become simply

$$\{\dot{F}_+ + \dot{F}_-\}/\dot{R} = 2(\sigma+P)^{-1} \{P\tilde{K} + (e_2^2-e_1^2)\ 8\pi R^4\} \quad , \tag{2.39}$$

$$\{\dot{F}_+ - \dot{F}_-\}/\dot{R} = 8\pi(P+\sigma/2) \quad . \tag{2.40}$$

§2.4 <u>Integration of the Equations of Motion.</u>

A first integral of (2.40) is obtained by noting that, for the intrinsic coordinates of (2.26), we have $u^i = (0,0,1)$, therefore $u^j_{;j} = 2\dot{R}/R$. From (2.36), it then follows that

$$(P+\sigma/2)\dot{R} = -\frac{1}{2}\, d(\sigma R)/d\tau \quad . \tag{2.41}$$

It is worth pointing out that (2.41) is just $\dot{M} = -P\dot{A}$, where A = area = $4\pi R^2$, and M is the total proper mass of the shell, defined by

$$M = 4\pi R^2\sigma \quad . \tag{2.42}$$

Using (2.41), (2.40) can be written equivalently as

$$\dot{F}_+ - \dot{F}_- = -4\pi d(\sigma R)/d\tau \quad , \tag{2.43}$$

which can be integrated to give (from (2.42))

$$F_+ - F_- = - M/R - C \quad , \tag{2.44}$$

where C is a constant of integration. Multiplying (2.44) by $(F_+ + F_-)$ and using (2.42) leads to

$$F_+ + F_- = 2(m_2-m_1)/R(4\pi R\sigma+C) - (e_2^2-e_1^2)/R^2(4\pi R\sigma+C) \quad . \tag{2.45}$$

From (2.5), a long but straightforward calculation gives

$$\tilde{K} = \tilde{K}_{ij}g^{ij} = \{F_+ + F_-\}/R + \frac{1}{2}\{\dot{F}_+ + \dot{F}_-\}/\dot{R} \quad . \tag{2.46}$$

Putting (2.46) into (2.39) and simplifying, yields

$$\{\dot{F}_+ + \dot{F}_-\} = (2P\dot{R}/\sigma R)\{F_+ + F_-\} + (e_2^2 - e_1^2)\dot{R}/4\pi R^4\sigma \quad . \tag{2.47}$$

Substituting (2.45) and its time derivative into (2.47), and comparing both sides, with the aid of (2.41), it follows that C must vanish in order that (2.45) be compatible with (2.39). Hence, using (2.42), equations (2.44) and (2.45) become

$$M = F_-R - F_+R = \{R^2(1+\dot{R}^2) - 2m_1R + e_1^2\}^{\frac{1}{2}} - \{R^2(1+\dot{R}^2) - 2m_2R + e_2^2\}^{\frac{1}{2}} , \tag{2.48}$$

$$F_-R + F_+R = \{2R(m_2-m_1) - (e_2^2-e_1^2)\}/M \quad . \tag{2.49}$$

Differentiation of (2.49), with the help of (2.41), will give (2.47), showing that, as expected, (2.49) is an integral of (2.39). Equations (2.48) and (2.49) can be combined, by adding and squaring, to give the final form, (2.1).

At this point, several remarks should be made concerning the integral of motion.

(i) This integral (2.1) along with (2.41) (i.e. along with $dM = -Pd(4\pi R^2)$) sums up the information contained in all the previous equations related to the motion of the system.

(ii) Our result is precisely that obtained by de la Cruz and Israel [46] for zero pressure, except that, in our case, the total proper mass M is not constant.

(iii) In the special case $m_1 = e_1 = 0$, considered by Kuchar [47], (2.1) can be written in the equivalent form

$$m = M(1+\dot{R}^2)^{\frac{1}{2}} - (M^2-e^2)/2R \quad , \tag{2.50}$$

expressing the conservation of total energy m of the shell. Our integral, then, is a generalization of this conservation equation, to non-zero m_1 and e_1.

(iv) The integral (2.1) is also a generalization of Kuchar's result in a different sense. In obtaining (2.50), Kuchar makes the assumption that the equation of state is polytropic, i.e. that the adiabatic exponent, defined by

$$\gamma = -\, d\ell nP/d\ell nA \quad , \tag{2.51}$$

is constant. In our considerations, however, no mention has been made of any restrictions on the equation of state, and, in fact, none need be made.

In Section 2.6 we consider the stability of this system, and we make use of definition (2.51), keeping in mind that γ is not constant, but is itself a function of the entropy and the density [48].

§2.5 Stability of an Uncharged Fluid Shell.

In this section, we avoid several involved computations, and consider the reasonably simple case of an uncharged fluid shell $(m_1 = e_1 = e_2 = 0)$. We want to study the stability of such a system against collapse.

The integral (2.1) can be simplified, in this case, to

$$m = M(1+\dot{R}^2)^{\frac{1}{2}} - M^2/2R = \text{total energy} \quad , \tag{2.52}$$

which, along with

$$dM = -\, Pd(4\pi R^2) \quad , \tag{2.53}$$

and the equation of state, completely describes the motion of the shell. Consider a momentarily static configuration, that is, a shell for which, at some instant, $\dot{R} = 0$. Taking $d/d\tau$ of (2.52), and using (2.53), it is immediate that

$$(M\ddot{R})_{\dot{R}=0} = 8\pi R(1-M/R)(P-\Gamma) \quad , \qquad (2.54)$$

where Γ is defined by

$$\Gamma = M^2/16\pi R^2(R-M) \quad , \qquad (2.55)$$

representing the gravitational self-attraction of the shell [49]. If the equations of state $P = P(\sigma_o)$, $\sigma = \sigma(\sigma_o)$ are given (σ_o is the "bare mass density"), then the configuration and initial acceleration of any momentarily static shell are completely determined by any two pieces of data. For example, suppose we are given the number of particles (i.e. given M_o) and the radius R . Then σ_o follows from $\sigma_o = M_o/4\pi R^2$, and hence $P = P(\sigma_o)$, $\sigma = \sigma(\sigma_o)$. M , m , and Γ are found from (2.42), (2.52) and (2.55) respectively. Finally $\ddot{R}$ follows from (2.54). Hence, to any momentarily static configuration of a given shell (of fixed M_o), we can associate a potential function $m(R,M_o)$:

$$m_{\dot{R}=0,M_o \text{ fixed}} = M - M^2/2R = M_o + U - M^2/2R \quad . \qquad (2.56)$$

In a quasistatic displacement, the change in this potential function can be written

$$dm_{\dot{R}=0,M_o \text{ fixed}} = - 8\pi R(1-M/R)(P-\Gamma)dR \quad ,$$

and since $1-M/R = (1-2m/R)^{1/2}$ from (2.56), we have

$$dm_{\dot{R}=0,M_o \text{fixed}} = -(1-2m/R)^{\frac{1}{2}} (P-\Gamma)d(4\pi R^2) \ . \tag{2.57}$$

This can be thought of as the energy change, measured by an observer at infinity. The term $(P-\Gamma)d(4\pi R^2)$ is the local measure of work done, ΔE, and it is decreased by $\sim\Delta Em/R$, the amount of energy lost in pushing ΔE up to the distant observer.

Suppose now that the shell is in equilibrium. Then

$$\dot{R} = \ddot{R} = 0 \quad , \tag{2.58}$$

hence from (2.54)

$$P = \Gamma \quad . \tag{2.59}$$

From (2.57), this is seen to be equivalent to

$$(dm/dR)_{\dot{R}=0,M_o \text{fixed}} = 0 \quad , \tag{2.60}$$

i.e. the potential energy is stationary for equilibrium.

As a result of this additional condition, we have that, given the functions $P(\sigma_o)$, $\sigma(\sigma_o)$, an equilibrium state is completely determined by a single datum. Under the assumption of equilibrium, $P = \Gamma$, equation (2.55), with $M = 4\pi R^2\sigma$, gives R as a function of P, σ:

$$R = P/\pi\sigma(\sigma + 4P) \quad . \tag{2.61}$$

Similarly

$$M = 4P^2/\pi\sigma(\sigma + 4P)^2 \qquad (2.62)$$

and

$$m = 4P^2(\sigma+2P)/\pi\sigma(\sigma+4P)^3 \qquad (2.63)$$

follow from (2.42) and (2.56).

Consider the class of all equilibrium states, each of which is specified by a particular equation of state, and one piece of data (say M_o) . In the non-relativistic limit $(P << \sigma_o)$, the equation of state is given by $P \alpha \sigma^2$ (see Appendix A), hence, from (2.63), $\lim_{P\to 0} m(P,\sigma(P)) = 0$. Similarly for the ultra-relativistic limit $(P >> \sigma_o)$, we have $P \alpha \sigma$ (see Appendix A) and $\lim_{P\to\infty} m(P,\sigma(P)) = 0$. We conclude that there must be at least one maximum for $m(P,\sigma)$. If we suppose, for simplicity, that there is exactly one maximum, we can represent the class of all equilibrium states $P = \Gamma$, by the solid curve in Figure 1. For a given m , then, there may be two or more equilibrium states. Presumably, not all of these will be stable against collapse.

Let us look at the stability of a particular shell of fixed M_o , whose equilibrium configuration is represented by the point E (Figure 1). The assignment of M_o completely determines m_E , R_E ,

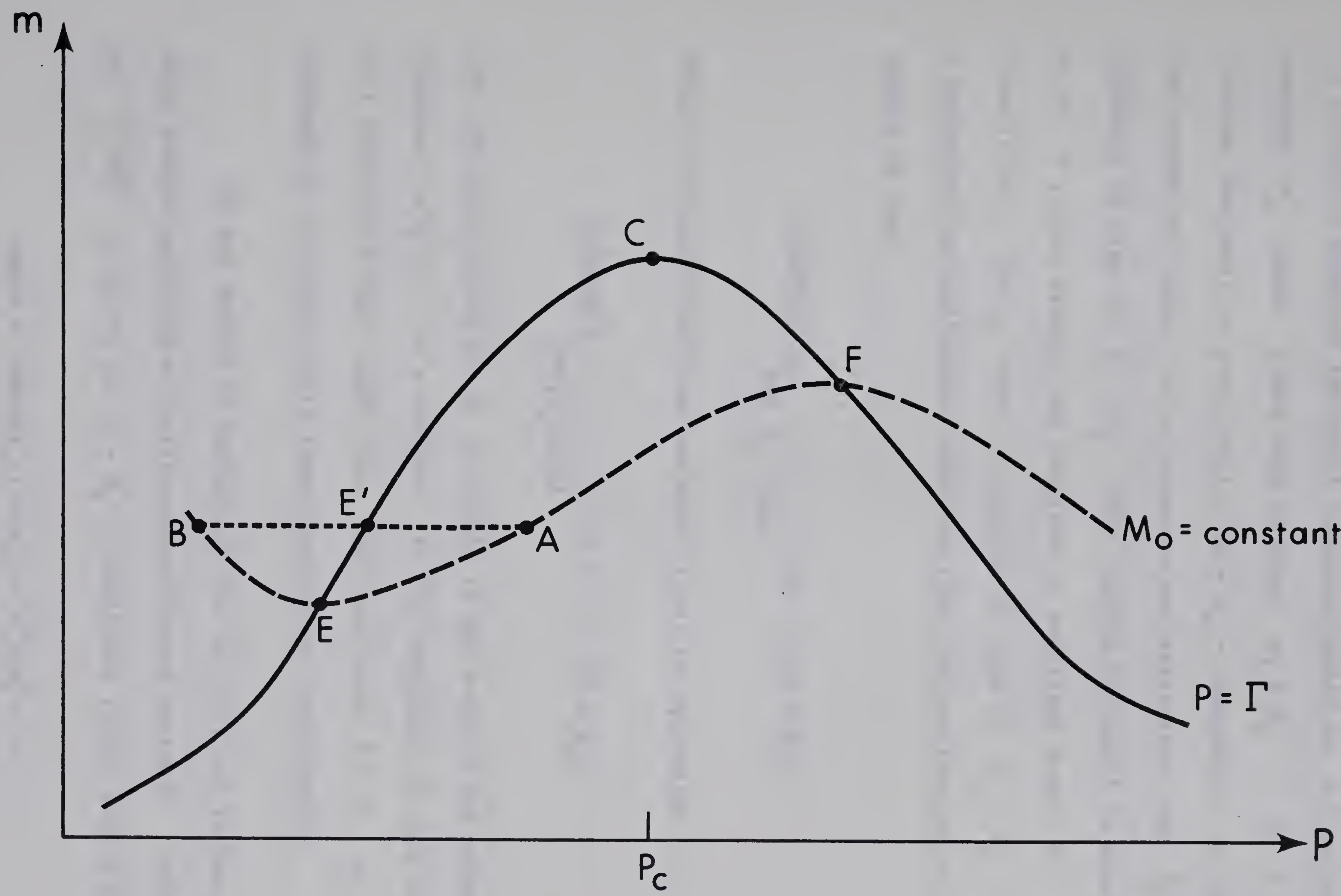
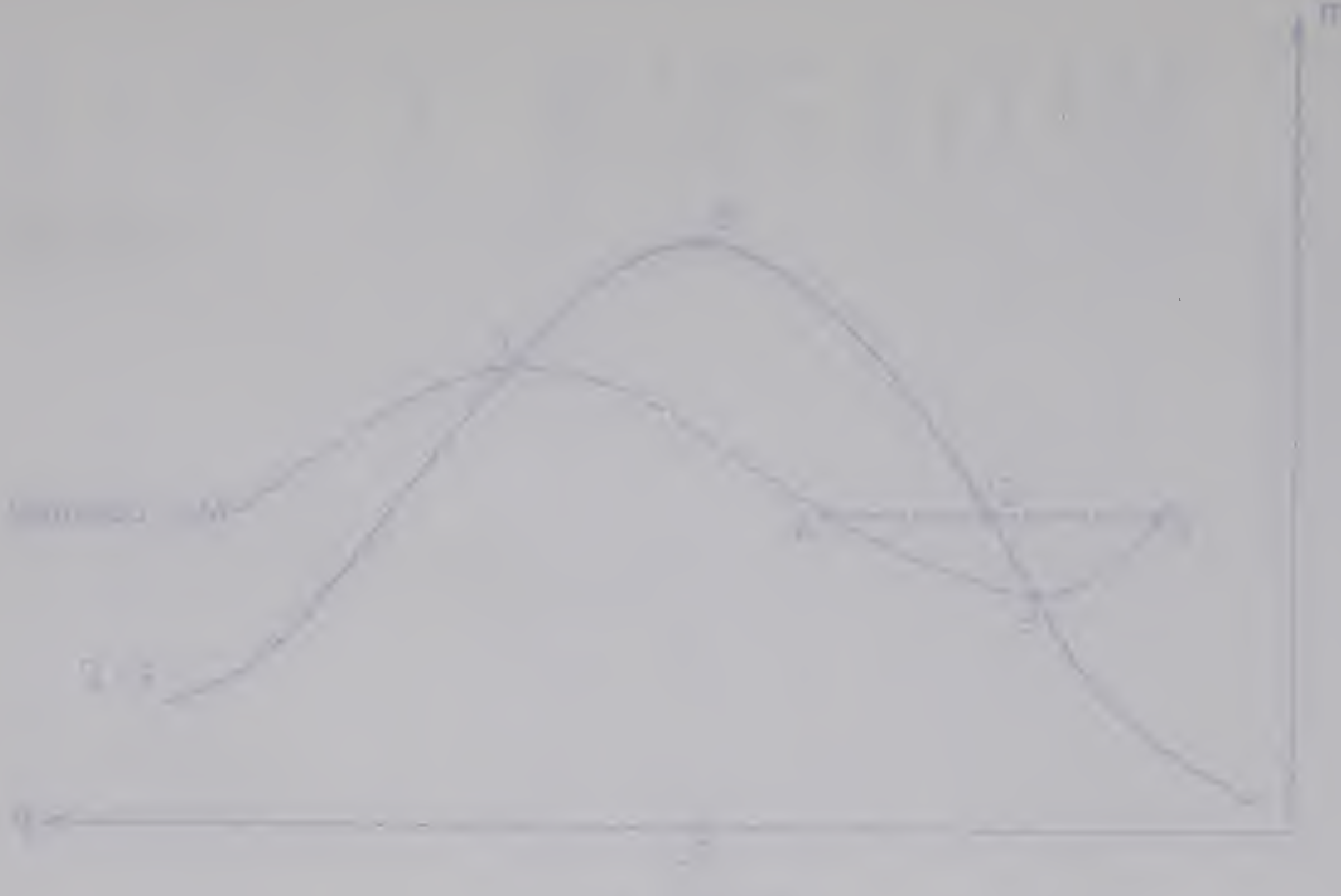


Figure 1

The (m,P) plane. Given the equation of state $P = P(\sigma)$, each point in this plane represents a momentarily static configuration (MSC) of some shell. All MSC's accessible to a shell containing a definite number of baryons (fixed M_o) lie on a curve in the (m,P) - plane (the dashed curve corresponds to one particular value of M_o). The curve $P = \Gamma$ represents all possible equilibrium states.



P_E . We now displace the shell quasistatically to A , as in (2.57), keeping M_o fixed. Since work is done, $m_A \neq m_E$. At A we release the shell to move freely $(m = m_A = \text{constant})$. In order that the system be stable, it is necessary and sufficient that $\ddot{R}_A$ is opposite in sign to $R_A - R_E$, thus causing the shell to move from A to B , turn around, and continue to oscillate about E' . Hence, from (2.54), $(P-\Gamma)_A$ must be opposite in sign to $R_A - R_E$. Therefore, a necessary and sufficient condition that the equilibrium at E be stable is that

$$(d/dR)(P-\Gamma)_{\dot{R}=0,M_o \text{ fixed}} < 0 \quad , \quad (R = R_E) \quad . \tag{2.64}$$

From (2.57) and (2.60), we see that (2.64) is equivalent to

$$(d^2m/dR^2)_{\dot{R}=0,M_o \text{ fixed}} > 0 \quad , \quad (R = R_E) \quad , \tag{2.65}$$

i.e. the equilibrium at E is stable if and only if the potential function $m(R_E,M_o)$ is a minimum. In the remainder of this section, we will understand that dm/dR etc. are taken with $\dot{R} = 0$, M_o fixed, although we no longer write these conditions, for brevity.

We now derive an expression for the critical radius, R_C , at which stable equilibrium configurations become unstable. Taking d/dR and d^2/dR^2 of (2.56) at $R = R_E$, conditions (2.60) and (2.65) become

$$dm/dR = (dM/dR)_E(1-M/R_E) + M^2/2R_E^2 = 0 \quad , \tag{2.66}$$

$$d^2m/dR^2 = (d^2M/dR^2)_E(1-M/R_E) - (dM/dR)_E((dM/dR)_E - 2M/R_E)/R_E - $$
$$- M^2/R_E^3 > 0 \quad . \tag{2.67}$$

From (2.51) and (2.53), it is not difficult to show that

$$(d^2M/dR^2)_E = (dM/dR)_E(1-2\gamma)/R_E \quad . \tag{2.68}$$

Substituting (2.68) into (2.67) and simplifying, by means of (2.66), the stability condition becomes

$$\gamma - 3/2 > \{M/(R_E-M)\}\ \{M/4(R_E-M) + 1\} \quad . \tag{2.69}$$

Using (2.56), (2.69) can be written

$$\gamma - \frac{3}{2} > \{-1 + R_E/(R_E^2-2R_Em)^{\frac{1}{2}}\}\ \{\tfrac{1}{4}(R_E/(R_E^2-2R_Em)^{\frac{1}{2}}) + \tfrac{3}{4}\} \quad . \tag{2.70}$$

To obtain an approximation, we assume $R_E >> m$ and neglect second order terms in m/R_E. Then (2.70) becomes simply the condition that the equilibrium at E is stable if and only if

$$R_E > m/(\gamma - \tfrac{3}{2}) \sim R_C \quad , \quad \tfrac{3}{2} < \gamma \leq 2 \quad \text{(see Appendix A)} \ . \tag{2.71}$$

It is interesting to note the similarity of this result and that of Chandrasekhar [50] for fluid spheres, namely that stable equilibrium occurs for

$$R_E > 2Km/(\gamma - \tfrac{4}{3}) \sim R_C \quad , \quad \tfrac{4}{3} < \gamma \leq \tfrac{5}{3} \quad , \quad K \sim 1 \quad .$$

We conclude this section by showing that equilibrium states such as E (where $P < P_C$) are stable against collapse, while configurations such as F $(P > P_C)$ are unstable. Since $dm/dP = (dm/dR) \times (dR/dP)$, we see from (2.59) that the M_o = constant curve (broken line in Figure 1) has extrema at E and F. It also follows from (2.60) that

$$d^2m/dP^2 = (d^2m/dR^2)(dR/dP)^2 \quad , \qquad (2.72)$$

hence from (2.65) the equilibrium at E is stable if and only if that curve has a minimum there. Since no extremum occurs between E and F, it is immediate that stability (instability) at E implies instability (stability) at F. This argument shows that the critical radius R_C must correspond to the equilibrium state at C. That the equilibrium is, in fact, stable at E, can be shown by considering (2.71) in the non-relativistic limit $(\gamma \to 2)$. In this limit, as E approaches the origin along $P = \Gamma$, we have $P \; \alpha \; \sigma^2$, hence $\lim_{P \to 0} m(P,\sigma(P)) = 0$, while $\lim_{P \to 0} R(P,\sigma(P))$ is finite, from (2.61). Therefore (2.71) is satisfied near the origin and we have stability there. Consider again a quasistatic displacement from E (near the origin) to A along M_o = constant. If we make the reasonable assumption that P is a monotone function of σ_o, then $dM_o = 0$ implies

$$(dR/dP) = - (R/2\sigma_o)(d\sigma_o/dP) < 0 \quad .$$

Thus, the increase in P from E to A is coincident with a contraction

of the shell. This fact, together with the stability at E, is sufficient to show $P > \Gamma$ throughout the region under the $P = \Gamma$ curve. It then follows that the equilibrium states are stable for $P < P_C$ and unstable for $P > P_C$.

§2.6 Stability of the Charged System.

We return now to the original system, and derive a condition for stability against collapse. We consider only configurations which are outside the upper Nordstrom horizon, i.e. we assume [51]

$$R \geq m_2 + (m_2^2 - e_2^2)^{\frac{1}{2}} > m_1 + (m_1^2 - e_1^2)^{\frac{1}{2}} .$$

It is convenient to deal with the integrals (2.48) and (2.49) rather than (2.1). For an equilibrium configuration at $R = R_o$ we must have (2.58) satisfied there. Then one equilibrium condition can be written, from (2.48) and (2.58), as

$$M = (F_-R - F_+R)_{R_o} = \{(R_o-m_1)^2 - (m_1^2-e_1^2)\}^{\frac{1}{2}} - \{(R_o-m_2)^2 - (m_2^2-e_2^2)\}^{\frac{1}{2}} . \tag{2.73}$$

Taking $d/d\tau$ of (2.48) and using (2.58), a second equilibrium condition is found to be

$$(dM/dR)_{R_o} = (R_o-m_1)(F_-R)^{-1}_{R_o} - (R_o-m_2)(F_+R)^{-1}_{R_o} . \tag{2.74}$$

Equivalently, if we write (2.49), from (2.48), in the form

$$m = m_2 - m_1 = F_- M - \{M^2 - (e_2^2 - e_1^2)\}/2R = \text{constant} \quad , \qquad (2.75)$$

and follow an argument identical to Section 2.5, we see that

$$(P-\Gamma)_{R_o} = 0 \quad , \qquad (2.76)$$

in order that equilibrium be maintained. In this case, Γ represents the gravitational self-attraction, together with the electrostatic forces, and is given by

$$\Gamma_{R_o} = \{M^2 - (e_2^2 - e_1^2) + 2M(m_1 R_o - e_1^2)(F_- R)^{-1}_{R_o}\}/16\pi R_o^2 \ \{(F_- R)_{R_o} - M\} \ . \qquad (2.77)$$

As in the previous section, the equilibrium will be stable, provided

$$(d/dR)(P-\Gamma)_{\dot{R}=0, M_o \text{ fixed}} < 0 \quad , \quad (R = R_o) \quad . \qquad (2.78)$$

If we now substitute for P from (2.53), and for Γ from (2.77), and evaluate the left side of (2.78), the resulting complicated expression can be reduced, with the aid of (2.73) and (2.74), to

$$(d^2M/dR^2)_{R_o} + \{(m_1^2 - e_1^2)(F_- R)^{-3}_{R_o} - (m_2^2 - e_2^2)(F_+ R)^{-3}_{R_o}\} > 0 \quad . \qquad (2.79)$$

From (2.68) and (2.74) we find

$$(d^2M/dR^2)_{R_o} = \{(R_o - m_1)(F_- R)^{-1}_{R_o} - (R_o - m_2)(F_+ R)^{-1}_{R_o}\}(1-2\gamma)/R_o \ . \qquad (2.80)$$

Substitution of (2.80) into (2.79) leads to the final form

$$\{(R_o-m_1)(F_-R)^{-1}_{R_o} - (R_o-m_2)(F_+R)^{-1}_{R_o}\} (1-2\gamma)/R_o > (m_2^2-e_2^2)(F_+R)^{-3}_{R_o} -$$

$$- (m_1^2-e_1^2)(F_-R)^{-3}_{R_o} \quad . \qquad (2.81)$$

This form of the stability condition is a function of the equilibrium radius R_o , alone, since m_1 , m_2 , e_1 , e_2 are fixed. Since $\gamma > \frac{3}{2}$, and $(dM/dR)_{R_o}$ is negative (from $dM = -Pd(4\pi R^2)$, $P > 0$) , it is clear, from (2.74), that the left side of (2.81) is positive. Thus, instability for some $R_o > m_2 + (m_2^2-e_2^2)^{1/2}$ can occur, only when the right side of (2.81) is positive. We now show that this is always the case.

The obvious restriction on the equation of state that $P > 0$, becomes, for equilibrium, the condition that $\Gamma > 0$. From (2.77), this restriction can be reduced to the simple condition

$$(m_1^2-e_1^2)/(R_o-m_1)^2 < (m_2^2-e_2^2)/(R_o-m_2)^2 \quad . \qquad (2.82)$$

This is a generalization of the condition $m^2 - e^2 > 0$, found by Kuchar [52]. Using (2.82), it is straightforward to show that the right side of (2.81) is always positive. We therefore see, under the physically reasonable assumptions $|e_1| < m_1$, $|e_2| < m_2$, that instability can occur outside the upper Nordstrom sphere. To show that, in fact, it does occur, it is sufficient to prove the existence of a critical radius $R_C > m_2 + (m_2^2-e_2^2)^{1/2}$. If we evaluate both sides of (2.81) at the Nordstrom sphere $R = m_2 + (m_2^2-e_2^2)^{1/2}$, we find that condition (2.81)

is violated, indicating instability. Similarly, it is trivial to show that, as $R \to \infty$, the condition is satisfied, hence instability sets in somewhere outside $R = m_2 + (m_2^2-e_2^2)^{1/2}$.

An approach which is perhaps more instructive, is to assume $R_o >> m_2$, and find an approximating stability condition, which is more informative than the exact relation. Neglecting second order terms in m_2/R_o , we can write (2.75), with $\dot{R} = 0$, as

$$m = M + \frac{1}{2} \{(m_1^2-e_1^2)/(R_o-m_1) - (m_2^2-e_2^2)/(R_o-m_2)\} ,$$

and hence an approximating condition can be found, from (2.65), to be

$$R_o > \frac{2\gamma-1}{2\gamma-3} \frac{m_2(m_2^2-e_2^2) - m_1(m_1^2-e_1^2)}{(m_2^2-e_2^2) - (m_1^2-e_1^2)} , \qquad (2.83)$$

for stability. Under the assumption $R_o >> m_2$, (2.82) becomes simply $m_1^2-e_1^2 < m_2^2-e_2^2$, i.e. the denominator of (2.83) is positive. Hence

$$R_o > \frac{2\gamma-1}{2\gamma-3} m_2 ,$$

for stability. In the case of particular interest, an effectively "hot" gas exerting pressure, we have $\gamma \to \frac{3}{2}$ and therefore, only very diffuse equilibrium states are stable.

CHAPTER III

Gravitational Collapse with Asymmetries

§3.1 Introduction

Every static nonspherical perturbation of Schwarzschild's exterior field, due to gravitational or electromagnetic sources within the stationary lightlike surface $g_{oo} = 0$ [53] becomes singular on this surface [54]. The same is true for all static zero-mass scalar perturbations [55] - spherical as well as nonspherical. Further, it appears from the discussion in §1.3, that stationary perturbations of Kerr's rotating solution have a similar property [56]. Assuming these results to be applicable to the asymptotically stationary exterior field of a collapsing star, one is led to the conjecture that all externally detectable asymmetries, including scalar and magnetic fields, must somehow decay, leaving behind Schwarzschild's vacuum field (or, in the case of nonvanishing charge and angular momentum, the "charged Kerr" field) as the sole external manifestation of the collapsed object.

To examine these questions, we carry out a dynamical analysis of two idealized collapse models, one involving a scalar monopole, the other a magnetic dipole. Our results support the foregoing conjecture and reveal the decay mechanism to be a rapid radiative leakage of the perturbing field, largely downwards through the event horizon.

We cast the Schwarzschild metric into the form

$$\left.\begin{aligned} ds^2 &= \alpha\, dxdy + r^2\, d\Omega^2 \quad , \\ \alpha &= 1 - \frac{1}{r} \quad , \end{aligned}\right\} \tag{3.1}$$

where the retarded and advanced time coordinates $-x,y$ are related to the standard Schwarzschild coordinates by

$$\left.\begin{aligned} x &= (r-1) + \ell n(r-1) - t \quad , \\ y &= (r-1) + \ell n(r-1) + t \quad . \end{aligned}\right\} \tag{3.2}$$

Lengths are measured in units of the Schwarzschild radius $2m = 1$.

Both of our models can be considered as linearly perturbed variations of the following basic situation (Figure 2). A thin hollow spherical shell of mass $m = \frac{1}{2}$ is initially static with radius $r = R_o >> 1$ [57]. At time $t = -\frac{1}{2} x_o \equiv -(R_o-1) - \ell n(R_o-1)$, it suddenly begins to collapse with the speed of light [58]. This model, adopted for mathematical simplicity, is highly artificial from an astrophysical point of view, but does not violate any of the principles of relativity theory. Moreover, our main interest is in the asymptotic behaviour of the exterior field as $t \to \infty$, and we do not expect this to depend too sensitively on the precise structure of the source or the initial conditions.

The perturbations we intend to impose on this idealized collapse picture are of two types - one a static spherically symmetric scalar monopole, the other a static axi-symmetric magnetic dipole. Both, however, are located at the centre of the collapsing shell, and both are assumed to be weak.

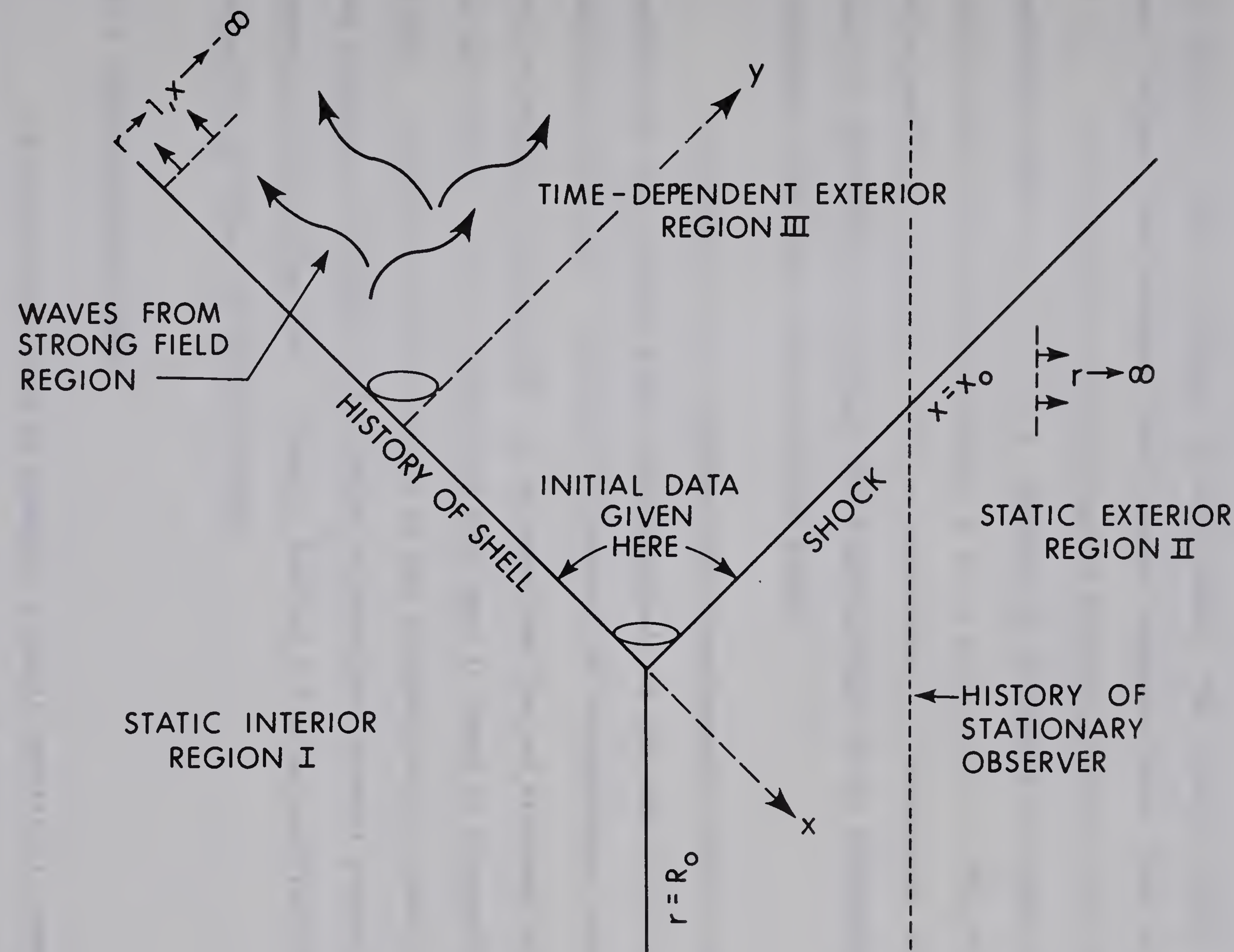


FIGURE 2

Space-time diagram for collapsing shell model.

Since news of the onset of collapse cannot reach the interior ahead of the shell itself, the initial static interior field (Region I in Figure 2) remains unchanged in both models. The exterior field, however, becomes time-dependent after passage through a shock front at $x = x_o$. Our problem is thus to find the perturbing field in the time-dependent region $x \leq x_o$, $y \geq 0$ (Region III in Figure 2), given Cauchy initial data on the characteristics $y = 0$, $x = x_o >> 1$. This we will do by numerical integration. (It is unnecessary to distinguish, at this level of approximation, between null hypersurfaces of the Schwarzschild background field and of the perturbed field).

In Section 3.2 we formulate the characteristic initial value problem for the collapse model just described, in the case of the scalar monopole perturbation. Section 3.3 contains the corresponding derivation for the magnetic dipole case. It is seen that the two problems can be put in very similar form - a convenient fact when it comes to integrating numerically. In Section 3.4 the results of the integrations for these two models are presented, including a discussion of where the external scalar or magnetic field energy goes. Finally, Section 3.5 gives qualitative interpretation and some concluding remarks.

§3.2 Scalar Monopole Model

For our first model we consider a static spherically symmetric zero-mass scalar monopole, located at the centre of the shell. It is assumed that this source is weak enough that any gravitational effects of the scalar energy density can be neglected for $r \geq 1$ (i.e. the scalar

field is <u>not</u> gravitationally coupled).

In order to find the perturbing scalar field in the time-dependent region, we must solve the scalar field equation

$$\left.\begin{aligned} \Box\Phi \equiv \frac{1}{\sqrt{-G}} (\sqrt{-G}\, g^{\alpha\beta}\, \Phi_{,\beta})_{,\alpha} = 0 \quad , \\ G \equiv \det g_{\alpha\beta} \quad , \end{aligned}\right\} \tag{3.3}$$

using the background metric (3.1) for that region [59]. From the spherical symmetry of Φ , (3.3) then becomes simply

$$\Phi_{xy} + \frac{\alpha}{2r} (\Phi_x + \Phi_y) = 0 \quad , \tag{3.4}$$

where the subscripts here denote partial differentiation.

It now remains to find the initial values on the characteristics $y = 0$, $x = x_o$. This we will do by finding exact solutions of (3.3) in both the static interior and static exterior regions (Region I and Region II of Figure 2, respectively), and using the fact that these solutions (but not their first derivatives) must match - not only with each other across their common boundary $(r = R_o)$, but also with the non-static solution across the characteristics which bound the time-dependent region.

First we solve (3.3) in the static interior region. In this case the background metric is flat [60], and (3.3) becomes, in the case of spherical symmetry,

$$r\, \Phi_{rr} + 2\, \Phi_r = 0 \quad ,$$

which has the general solution

$$\Phi = c_1 + c_2/r \quad . \tag{3.5}$$

In the static exterior region, we have the Schwarzschild background (1.1), and for static spherical symmetry, with $2m = 1$, (3.3) becomes

$$(r-1)\ \Phi_{rr} + 2(1 - \frac{1}{2r})\ \Phi_r = 0 \ .$$

The solution in this case [61] is

$$\Phi = c_3 + c_4\ \ell n\ (\frac{r-1}{r}) \quad .$$

The arbitrary constant in Φ makes no difference to the field, so we choose $c_3 = 0$ for simplicity, and hence

$$\Phi = c_4\ \ell n\ (\frac{r-1}{r}) \quad . \tag{3.6}$$

At this point it is convenient to assume the initial radius of the shell is very large $(R_o >> 1)$. Then throughout the static exterior region $(r \geq R_o >> 1)$ we have

$$\ell n(\frac{r-1}{r}) \approx - \frac{1}{r} \quad . \tag{3.7}$$

In particular, in order that (3.5) and (3.6) match up across $r = R_o >> 1$, it follows from (3.7) that

$$c_1 + c_2/R_o \approx - c_4/R_o \quad .$$

This tells us that

$$c_1 = 0 \quad ; \quad c_2 = - c_4 = c \quad . \tag{3.8}$$

From (3.8) our static interior solution is therefore

$$\Phi = \frac{c}{r} \quad , \tag{3.9}$$

and from (3.7) and (3.8) the static exterior solution is

$$\Phi = - c \, \ell n(\frac{r-1}{r}) \approx \frac{c}{r} \quad . \tag{3.10}$$

The value of Φ on the characteristic $y = 0$ (history of the collapsing shell) is then determined by the solution (3.9), and so (choosing $c = 1$ for simplicity) we have

$$\Phi(x,y=0) = \frac{1}{r(x,y=0)} \quad . \tag{3.11}$$

Similarly, for the characteristic $x = x_o = 2(R_o-1) + 2 \, \ell n(R_o-1) >> 1$ (history of the shock front) we use (3.10) to conclude that

$$\Phi(x=\infty,y) = \frac{1}{r(x=\infty,y)} \quad . \tag{3.12}$$

If we define $\psi \equiv r \, \Phi$, the characteristic initial value problem given by (3.4), (3.11) and (3.12) takes the simple form

$$\psi_{xy} = \frac{\alpha}{4r^3} \psi \quad , \tag{3.13}$$

$$\left.\begin{aligned} \psi(x,0) &= 1 \quad , \\ \psi(\infty,y) &= 1 \quad . \end{aligned}\right\} \tag{3.14}$$

§3.3 Magnetic Dipole Model

We now turn to our second model in which a static axi-symmetric magnetic dipole of moment μ is placed at the centre of the shell. As before, it is assumed that $\mu^2 << 1$, i.e. the perturbing field is weak enough that it in no way affects the geometry of the system.

Our first step is to write down the source-free electromagnetic field equations

$$F^{\mu\nu}{}_{|\nu} = 0 \quad , \tag{3.15}$$

$$F_{[\mu\nu,\sigma]} = 0 \quad , \tag{3.16}$$

where the electromagnetic field tensor is given by

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \quad , \tag{3.17}$$

and $A_\nu = (A_o, A_1, A_2, A_3)$ is the associated potential 4-vector. We need concern ourselves only with (3.15), since (3.16) is satisfied identically by virtue of (3.17).

As in §3.2, we want to solve these field equations in the time-dependent region, given initial values on the characteristics $y = 0$, $x = x_o$. If we write out the equations (3.15) on the Schwarzschild background (3.1) and use (3.17), together with the facts that the field is axi-symmetric and gauge invariant, we obtain the following results (see Appendix B):

$$\underline{\mu = 3}: \quad \frac{2}{\sin\theta} A_{3,01} + \frac{\alpha}{2r^2}\left(\frac{A_{3,2}}{\sin\theta}\right)_{,2} = 0 \quad , \tag{3.18}$$

$$\underline{\mu = 2}: \quad A_2 = f(x,\theta) + g(y,\theta) \;,\quad f \text{ and } g \text{ arbitrary}, \tag{3.19}$$

$$\underline{\mu = 1}: \quad 4\left(\frac{r^2}{\alpha} A_{0,1}\right)_{,0} + \frac{1}{\sin\theta}(\sin\theta\, A_{0,2})_{,2} = \frac{1}{\sin\theta}(\sin\theta\, A_{2,0})_{,2} \quad , \tag{3.20}$$

$$\underline{\mu = 0}: \quad 4\left(\frac{r^2}{\alpha} A_{1,0}\right)_{,1} + \frac{1}{\sin\theta}(\sin\theta\, A_{1,2})_{,2} = \frac{1}{\sin\theta}(\sin\theta\, A_{2,1})_{,2} \quad . \tag{3.21}$$

As before, in order to obtain suitable initial conditions, we must find corresponding solutions of the field equations for the static interior and static exterior regions. Because these solutions are static, it is convenient to use the standard Schwarzschild coordinates $\bar{x}^\nu = (t,r,\theta,\phi)$ which are related to our original $x^\nu = (x,y,\theta,\phi)$ by (3.2). The corresponding components of the potential 4-vector are then given by

$$\bar{A}_\nu = A_\mu \frac{\partial x^\mu}{\partial \bar{x}^\nu} \quad ,$$

so that

$$\left.\begin{aligned} \bar{A}_o &= 2(A_1 - A_o) \quad , \quad \bar{A}_1 = \frac{2}{\alpha}(A_o + A_1) \quad , \\ \bar{A}_2 &= A_2 \quad , \quad \bar{A}_3 = A_3 \quad . \end{aligned}\right\} \tag{3.22}$$

Consider first the static interior region where space-time is flat $(\alpha=1)$. From the Maxwell equations with magnetic field $\underset{\sim}{H}$ and vanishing electric field, we can write

$$\operatorname{div} \underset{\sim}{H} = 0 \quad , \tag{3.23}$$

from which it follows that

$$\underset{\sim}{H} = \operatorname{curl} \underset{\sim}{A} = \operatorname{grad} \Omega \quad , \tag{3.24}$$

where $\underset{\sim}{A} = (\bar{A}_1, \bar{A}_2, \bar{A}_3)$ is the 3-vector potential. By virtue of (3.23) and (3.24) we have

$$\nabla^2 \Omega = 0 \quad ,$$

which, for a magnetic dipole of moment μ, yields a solution

$$\Omega = - \frac{\mu \cos \theta}{r^2} \quad . \tag{3.25}$$

Taking contravariant components of (3.24) gives

$$g^{-\frac{1}{2}} e^{ipq} \bar{A}_{q,p} = g^{ij} \Omega_{,j} \quad , \tag{3.26}$$

where $g = \det g_{ij}$ and e^{ipq} is the standard permutation symbol. Using (3.25), (3.26) has, as one of its solutions

$$\bar{A}_1 = \bar{A}_2 = 0 \quad , \quad \bar{A}_3 = \mu \sin^2\theta / r \quad . \tag{3.27}$$

(In addition, $\bar{A}_o$ can be chosen zero since there is no electric field). Returning to our original coordinates x^ν, a solution of the field equations in the static interior region is now found, from (3.22), and (3.27), to be

$$A_o = A_1 = A_2 = 0 \quad , \quad A_3 = \frac{\mu \sin^2\theta}{r} \quad . \tag{3.28}$$

One can easily check that this solution satisfies (3.18) - (3.21) with $\alpha=1$, as required.

Turning to the static exterior region, we must find the corresponding static solution. It is clear that, in this region as well, we may choose

$$A_o = A_1 = A_2 = 0 \quad , \tag{3.29}$$

so that (3.19) - (3.21) are trivially satisfied. Let us define

$$A_3 = \psi(x,y) \sin^2\theta \quad . \tag{3.30}$$

(Comparison with (3.28) shows that $\psi = \mu/r$ represents the appropriate static interior solution.) From (3.30), we can reduce (3.18) to get

$$\psi_{xy} = \frac{\alpha}{2r^2} \psi \quad , \tag{3.31}$$

where again the subscripts denote partial differentiation.

For a static solution, it is easily shown from (3.2) that

$$\psi_{xy} = \frac{\alpha^2}{4} \psi_{rr} + \frac{\alpha}{4r^2} \psi_r \quad . \tag{3.32}$$

Combining (3.31) and (3.32) we obtain

$$\alpha \psi_{rr} + \frac{1}{r^2} \psi_r = \frac{2}{r^2} \psi \quad ,$$

which can be solved to give

$$\psi = C\, r^2 \int \frac{dr}{r^3(r-1)} \quad , \quad C \text{ arbitrary constant.} \tag{3.33}$$

If we assume as before that $R_o >> 1$, then $r >> 1$ throughout the static exterior region, and (3.33) becomes

$$\psi \approx \frac{\mu}{r} \quad , \tag{3.34}$$

with suitable choice for C . This result is therefore identical to the static interior solution.

It remains to find the manner in which these two solutions join up with the non-static solution across the characteristics $y = 0$, $x = x_o >> 1$. As it turns out, the jump conditions for the electromagnetic field require continuity of the A_ν across the characteristic surfaces (see Appendix C). In the time-dependent region, then, we can again take $A_o = A_1 = A_2 = 0$, so that the characteristic initial value problem is given simply by

$$\psi_{xy} = \frac{\alpha}{2r^2}\,\psi \quad , \tag{3,35}$$

$$\left.\begin{aligned} \psi(x,0) &= \frac{\mu}{r(x,0)} \quad , \\ \psi(\infty,y) &= \frac{\mu}{r(\infty,y)} = 0 \quad . \end{aligned}\right\} \tag{3.36}$$

Comparison with the scalar monopole case, (3.13) - (3.14), shows that these two collapse models are strikingly similar [62].

§3.4 Results of Numerical Integration

The two characteristic initial value problems, (3.13) - (3.14) and (3.35) - (3.36), can now be integrated numerically over the time-dependent region $-\infty < x < \infty$, $0 \leq y < \infty$ [63]. Some results of these integrations can be seen in Figure 3 (scalar monopole perturbations) and Figure 4 (magnetic dipole perturbations).

Several general remarks apply to both cases. To a stationary external observer $(r = \text{const.})$ the field appears nearly constant for a period about equal to the Newtonian free-fall time [64] i.e. down to $x \approx 0$. The epoch $x \approx 0$ is marked by the fairly sudden onset of an oscillatory decline towards zero. On the horizon $r = 1$ itself, the field displays a similar damped oscillatory behaviour as a function of y . A free-falling observer close to the collapsing body sees no decline in the field, but we can find no support (at least in these idealized models) for a suggestion by Ginzburg [65] based on a quasi-static analysis for a magnetic dipole perturbation, that the field becomes infinitely compressed against the body.

With the knowledge that, in both models, the external field decays in an oscillatory fashion, it is natural to ask what becomes of the external scalar and magnetic field energies. We can answer this question by making use of a certain identity, which we now derive.

From (3.2) it is immediate that both the scalar and magnetic equations, (3.13) and (3.35), are of the same form, namely

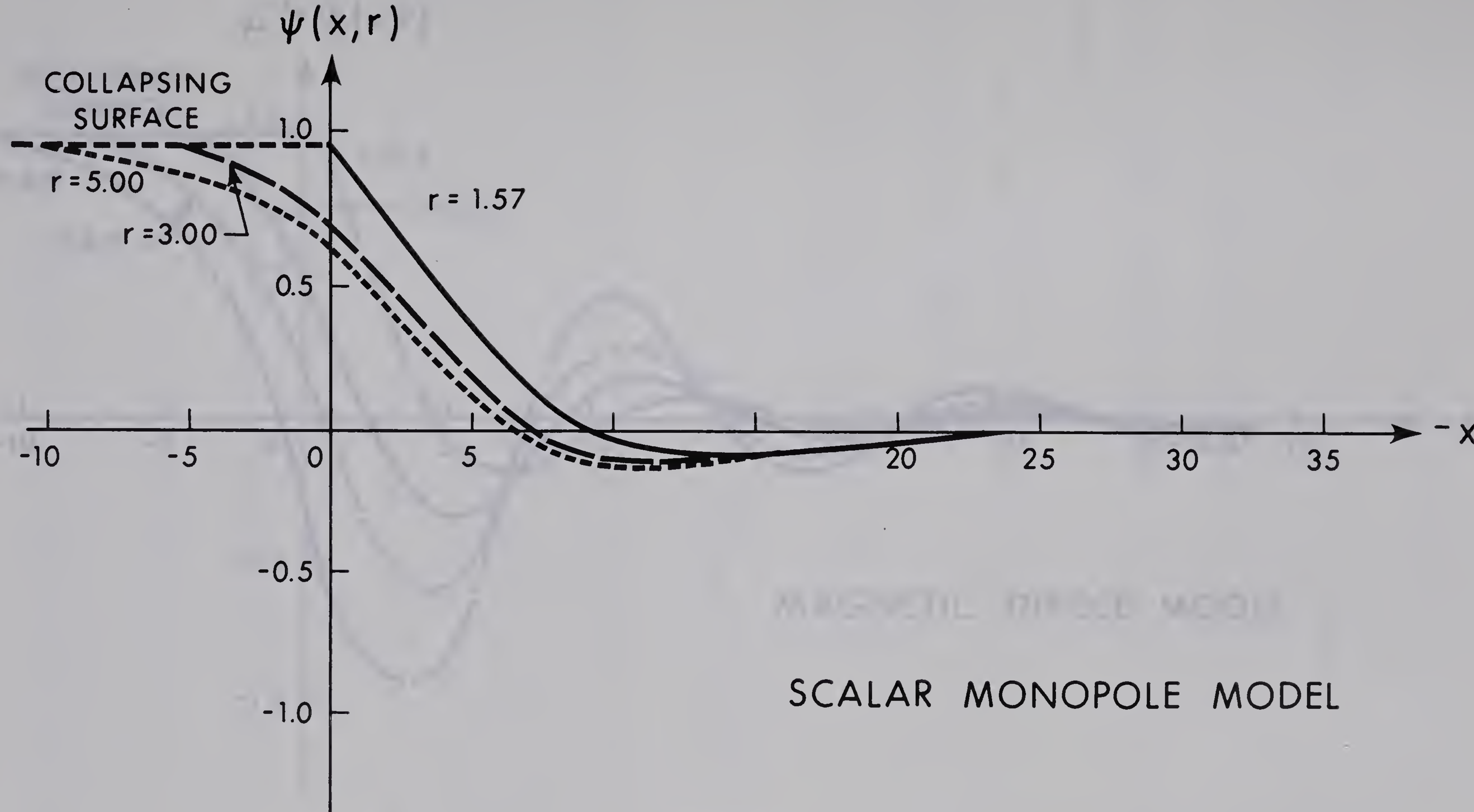


FIGURE 3

ψ as a function of retarded time $(-x)$ for stationary observers with radial coordinates equal to 1.57, 3, and 5 Schwarzschild radii. The scalar potential $\Phi = \psi/r$.

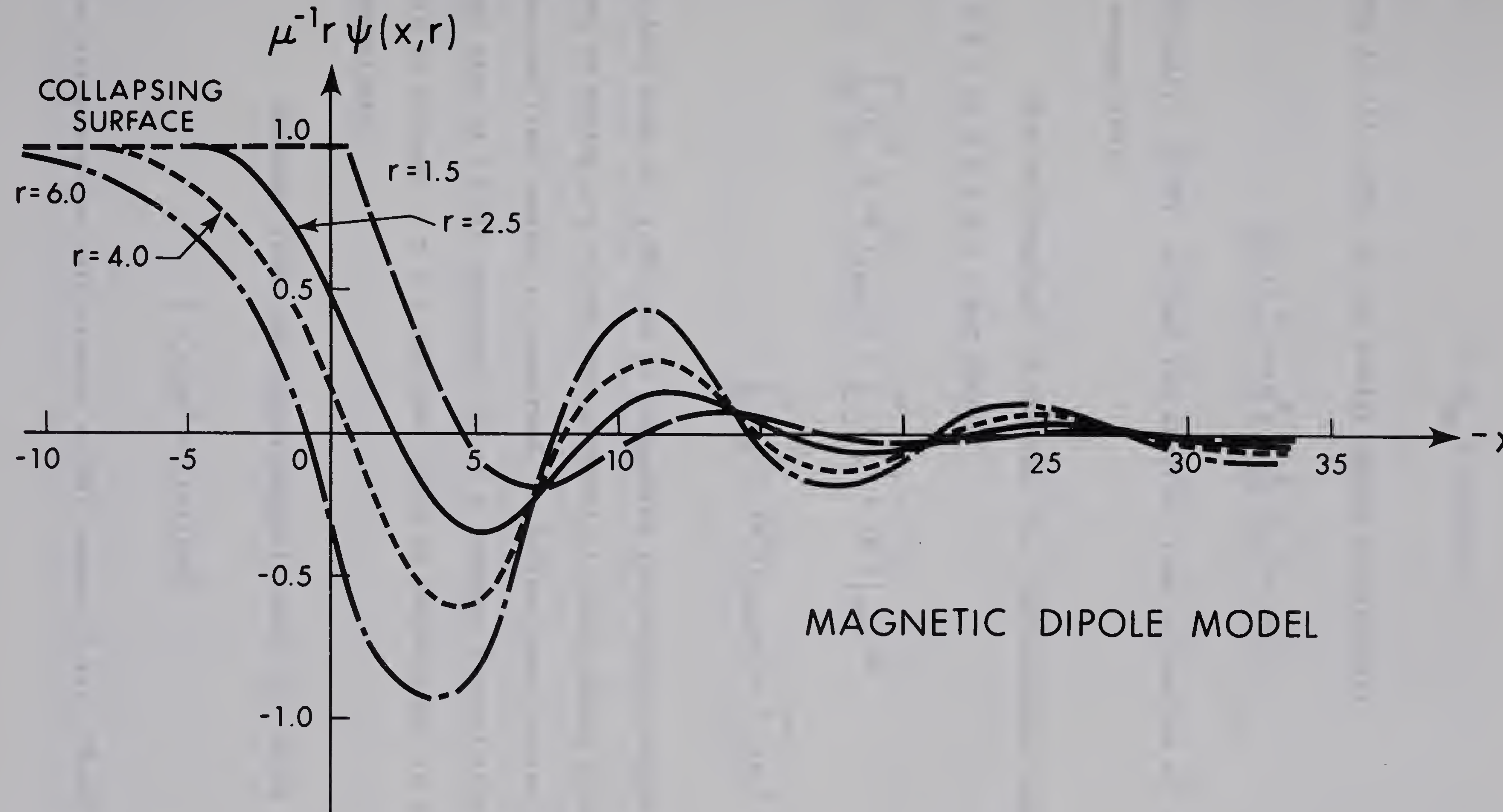


FIGURE 4

$r\psi$ as a function of retarded time $(-x)$ for stationary observers with radial coordinates equal to 1.5, 2.5, 4, and 6 Schwarzschild radii. The vector potential $\underset{\sim}{A} = \underset{\sim}{e}_{(\phi)} r^{-1} \psi \sin\theta$, where $\underset{\sim}{e}_{(\phi)}$ is a unit azimuthal vector.

$$\psi_{xy} = f(x+y)\,\psi \,. \tag{3.37}$$

Using (3.37), then, we can easily obtain the identity

$$\frac{\partial}{\partial x}\,({\psi_y}^2 + f\psi^2) \equiv \frac{\partial}{\partial y}\,({\psi_x}^2 + f\psi^2) \quad . \tag{3.38}$$

Physically, (3.38) is just the explicit form of the law of conservation of energy [66].

Now we integrate (3.38) over the time-dependent region $-\infty < x < \infty$, $0 \leq y < \infty$, and obtain

$$\begin{aligned} &\int_0^\infty ({\psi_y}^2 + f\psi^2)\Big|_{x=\infty} dy + \int_{-\infty}^\infty ({\psi_x}^2 + f\psi^2)\Big|_{y=0} dx \\ &\qquad = \int_0^\infty ({\psi_y}^2 + f\psi^2)\Big|_{x=-\infty} dy + \int_{-\infty}^\infty ({\psi_x}^2 + f\psi^2)\Big|_{y=\infty} dx \,. \end{aligned} \tag{3.39}$$

The left side of (3.39) can be integrated for both the scalar and magnetic cases by making use of the characteristic initial values (3.14) or (3.36). Since the field energy is conserved, it follows that the two terms on the right side of (3.39) represent (up to a constant factor) that energy which falls in through $r = 1$, and that which is radiated out to infinity, respectively.

Consider first the scalar monopole case, for which

$$f = \alpha/4r^3 = (r-1)/4r^4 \quad . \tag{3.40}$$

If we integrate the left side of (3.39), using (3.14), (3.40) and the fact

that $dx = dy = 2\ dr/\alpha$ from (3.2), we obtain the value $1/4$. In addition, the contribution of the terms $f\psi^2$ in both the right hand integrals of (3.39) is zero. For the first term, this is true because $x = -\infty$ implies $r = 1$, by (3.2), and hence $f = 0$, from (3.40). For the second term, (3.40) again guarantees that f vanishes, since $r = \infty$ when $y = \infty$ [67]. Thus (3.39) becomes

$$\int_0^\infty {\psi_y}^2\Big|_{x=-\infty} dy + \int_{-\infty} {\psi_x}^2\Big|_{y=\infty} dx = \frac{1}{4} . \tag{3.41}$$

The two remaining integrals in (3.41) can be evaluated numerically, with data provided by the computer output from the original integrations. The result of these calculations is

$$\int_0^\infty {\psi_y}^2\Big|_{x=-\infty} dy \approx .168 \quad , \quad \int_{-\infty}^\infty {\psi_x}^2\Big|_{y=\infty} dx \approx .080 . \tag{3.42}$$

Since the sum of these integrals is very close to $\frac{1}{4}$, almost all of the scalar field energy (i.e. 99%) is accounted for. We may therefore say with confidence that about two-thirds (approximately 67%) of the energy falls in through the horizon, while the remainder is radiated away to infinity.

We now turn to the magnetic dipole model. This time

$$f = \frac{\alpha}{2r^2} = \frac{(r-1)}{2r^3} . \tag{3.43}$$

Using (3.36) and (3.43), the same approach as before yields the value $3\mu^2/8$ for the left side of (3.39). Again the contribution of $f\psi^2$ to both of the remaining integrals is zero, from (3.43), so that (3.39)

reduces to

$$\int_0^\infty \psi_y^{\ 2}\Big|_{x=-\infty} dy + \int_{-\infty}^\infty \psi_x^{\ 2}\Big|_{y=\infty} dx = \frac{3\mu^2}{8} \ . \qquad (3.44)$$

Numerically integrating the left side of (3.44) gives

$$\int_0^\infty \psi_y^{\ 2}\Big|_{x=-\infty} dy \approx .360\ \mu^2 \ , \quad \int_{-\infty}^\infty \psi_x^{\ 2}\Big|_{y=\infty} dx \approx .015\mu^2 \ . \qquad (3.45)$$

Here, virtually all of the electromagnetic field energy is accounted for, and the corresponding proportions which pass in through the Schwarzschild radius and out to infinity are 96% and 4%, respectively [68].

§3.5 Discussion and Remarks

In the previous three sections we have considered two highly idealized models of perturbed collapse, involving weak scalar monopole and magnetic dipole fields anchored to thin spherical shells collapsing with the speed of light. We have seen that the perturbing field decays in an oscillatory manner, with most of the field energy falling into the "black hole" formed when the object collapses inside $r = 1$. A summary of these results appears in Table 1.

We conclude this section with several remarks worth noting.

(i) In the electromagnetic model, only 4% of the field energy escapes to the outside, as compared with 33% for the scalar model. The difference here is presumably due to the fact that scalar fields can emit monopole radiation, while electromagnetic fields cannot.

	Scalar Monopole Model	Magnetic Dipole Model
Type of Field	Weak scalar monopole at centre of shell with scalar potential Φ	Weak magnetic dipole at centre of shell with potential 4-vector $A_o = A_1 = A_2 = 0$, $A_3 = \psi \sin^2\theta$.
Partial Differential Equation (to be integrated numerically over time-dependent region)	$\psi_{xy} = f(r)\,\psi$, $f(r) = \alpha/4r^3$, $\psi = r\,\Phi$	$\psi_{xy} = f(r)\,\psi$, $f(r) = \alpha/2r^2$
Characteristic Initial Values	$\psi(x,0) = 1$, $\psi(\infty,y) = 1$	$\psi(x,0) = \mu/r(x,0)$, $\psi(\infty,y) = \mu/r(\infty,y) = 0$.
Outcome of Numerical Integration	External field undergoes rapid oscillatory decay (Figure 3).	External field undergoes rapid oscillatory decay (Figure 4).
Conservation Identity	$\frac{\partial}{\partial x}(\psi_y^{\,2} + f\psi^2) = \frac{\partial}{\partial y}(\psi_x^{\,2} + f\psi^2)$	$\frac{\partial}{\partial x}(\psi_y^{\,2} + f\psi^2) = \frac{\partial}{\partial y}(\psi_x^{\,2} + f\psi^2)$
Proportion of Field Energy		
(i) escaping to infinity	approximately 33%	approximately 4%
(ii) falling inside $r = 1$	approximately 67%	approximately 96%

TABLE I: Summary of Analysis for Two Idealized Models of Perturbed Gravitational Collapse

(ii) Israel [69] has described the oscillatory nature of the decay as a "bathplug effect": not all of the inward-falling field energy is sucked into the hole at once; instead, some of it is turned aside by field pressure, and it swirls about the hole on a time-scale several times larger than the Schwarzschild characteristic time. In this same connection, Price's analysis [70] establishes the existence of a curvature-induced potential barrier at $r \approx 3m$ which prevents the escape of low-frequency waves to infinity. Further, the potential barrier causes the low-frequency waves to oscillate - once for monopole fields (ℓ=0), twice for dipole fields (ℓ=1), etc. [71].

(iii) With regard to the second (magnetic) model, our results have an important bearing on the question of observability of collapsed stars, and their distinguishability from neutron stars. Unlike neutron stars, which can remain intensely magnetized for periods exceeding 10^6 years [72], a collapsed object cannot be a source of synchrotron or other radiation requiring the agency of magnetic fields. However, accretion of interstellar material in a favourable environment, e.g. in close binary systems, could produce a strong source of thermal x-ray bremsstrahlung.

(iv) Again, concerning the magnetic dipole model, a glance at Figure 4 reveals a curious fact. The r = const. curves appear to have several points of coincidence. This has been further verified by plotting similar curves (not shown) for several additional values of r, with the same result. We have found no explanation for this,

but it may possibly have significance regarding a resonant quality of space-time [73].

(v) A similar analysis for the case of a weak gravitational quadrupole perturbation has been carried out by de la Cruz and Israel [74]. The results lack the completeness and accuracy which our models yield, but they are qualitatively alike. In particular, only a small amount of gravitational energy escapes to infinity.

(vi) Our efforts to find the asymptotic behaviour of the external field have unfortunately failed. Price [75], however, does succeed in showing how the tails of the radiated waves die out for large t. He considers weak, zero-mass fields of arbitrary integer spin s on a Schwarzschild background, and shows that, as $t \to \infty$, a multipole moment of order ℓ decays in time as $\ln t/t^{2\ell+2}$ $(\ell \geq s > 0)$ or $1/t^2$ $(\ell = s = 0)$ [76].

CHAPTER IV

Event Horizons in Static Scalar-Vacuum Space-Times

§4.1 Introduction

Recent interest in the theory of gravitational collapse has raised many questions regarding the existence and nature of event horizons in relativity. Some definite results are known. Israel has shown [77] that for the class of asymptotically flat, static vacuum fields, only the spherically symmetric Schwarzschild solutions with $m \geq 0$ have a regular event horizon $(r = 2m)$, and [78] that for the corresponding electrovac space-times, the Reissner-Nordström solutions with $m \geq G^{\frac{1}{2}}|e|/c$ are the only ones with non-singular horizons. In view of these special cases, it is therefore natural to ask whether, for arbitrary, asymptotically flat static fields, a regular event horizon is destroyed by any asymmetric perturbation due to sources within the surface $g_{oo} = 0$.

In this connection there has been some recent interest in another special class - namely, the coupled gravitational and massless scalar fields (where by "massless scalar field" we mean a scalar field for zero-mass particles). The spherically symmetric solution of Janis, Newman and Winicour (JNW) [79] has the interesting property that the event horizon is a singular point in the space no matter how small the coupling constant becomes. Penney [80] has suggested that this surprising result is due to the imposition of spherical symmetry, and that, by considering asymmetric solutions, one is led to a non-singular horizon. However, his

example in support of this contention contains an error [81], and, in fact, his solution is singular at the horizon.

In this paper we propose to clear up much of the controversy about event horizons associated with asymptotically flat, static, massless scalar fields interacting with Einstein fields. Our main result can be stated as a theorem: every zero-mass scalar field which is gravitationally coupled, static and asymptotically flat, becomes singular at a simply-connected event horizon. This theorem immediately obviates Penney's search for a non-singular asymmetric horizon.

We proceed by reformulating the given conditions in terms of the geometry of the surfaces g_{oo} = const. (§§4.2 - 4.3). The theorem is stated in detail (§4.4) and proved (§4.5). In the special case where we neglect the gravitational effect of the scalar energy density, the solutions are computed explicitely (§4.6). In §4.7, we discuss some properties of the singular horizons.

§4.2 <u>Static Fields</u>

This section deals with the general static field. We want to reformulate the Einstein equations as conditions on the geometry of the equipotential surfaces [82].

The signature of the metric is $-+++$. Capitalized Latin indices run from 0 to 3. Three-dimensional and two-dimensional subtensors are distinguished by Greek indices (range 1-3) and by lower-case Latin indices (range 2-3). Covariant differentiation with respect

to the 4-dimensional, 3-dimensional, and 2-dimensional metrics is denoted by ∇, a stroke, and a semicolon, respectively.

A space-time is called "static" if it admits a regular, hypersurface-orthogonal, Killing vector field ξ, which is time-like $(\xi_A \xi^A < 0)$ over some domain; i.e., we have

$$\nabla_A \xi_B + \nabla_B \xi_A = 0 \qquad \text{(Killing's equations)}, \tag{4.1}$$

$$\xi_{[A} \nabla_C \xi_{B]} = 0 \qquad \text{(hypersurface-orthogonality)}. \tag{4.2}$$

From (4.1) and (4.2) it is easy to show that

$$\partial_{[A}(V^{-2} \xi_{B]}) = 0 \ , \tag{4.3}$$

where we define V by

$$V \equiv (-\xi_A \xi^A)^{\frac{1}{2}} \ . \tag{4.4}$$

Therefore, throughout a simply-connected domain in which ξ is time-like, (4.3) enables us to choose $t(x^A)$ such that

$$V^{-2} \xi_A = -\partial_A t \ . \tag{4.5}$$

This allows us to introduce "static coordinates" $x^o = t$, x^α, where the latter are any three independent solutions of

$$\xi^A \partial_A x^\alpha = 0 \ . \tag{4.6}$$

Using (4.1), (4.4), (4.5) and (4.6), it is straightforward to show that, in the domain where $\xi_A \xi^A < 0$, the metric can be put in the form

$$\left.\begin{aligned} ds^2 &= g_{\alpha\beta}(x^1,x^2,x^3)dx^\alpha dx^\beta - V^2dt^2 \ , \\ V &= V(x^1,x^2,x^3) > 0 \ . \end{aligned}\right\} \tag{4.7}$$

We can further decompose the metric form (4.7) if we define

$$\rho^{-1} \equiv (V_{|\alpha}\, V^{|\alpha})^{\frac{1}{2}} \ , \tag{4.8}$$

and assume ρ^{-1} vanishes nowhere in the domain of interest [83]. As intrinsic coordinates for the equipotential 2-spaces V = const., t = const., we introduce functions θ^1, θ^2 which are constant along the orthogonal trajectories ($g^{\alpha\beta}\partial_\alpha\theta^a\partial_\beta V = 0$). The spatial metric then becomes

$$g_{\alpha\beta}dx^\alpha dx^\beta = g_{ab}(V,\theta)d\theta^a d\theta^b + [\rho(V,\theta)]^2\, dV^2 \ . \tag{4.9}$$

Suppose $\underset{\sim}{n}$ is the unit space-like vector normal to the equipotential surfaces. Then, from (4.8),

$$n_\alpha = \rho\partial_\alpha V = \rho^{-1}\partial x^\alpha(V,\theta)/\partial V \ . \tag{4.10}$$

In addition, we let $\underset{\sim}{e}_{(a)}$ be the tangential base vectors associated with θ^a, so that

$$e^\alpha_{(a)} = \partial x^\alpha(V,\theta)/\partial\theta^a, \quad e^{(a)}_\alpha \equiv g^{ab}e_{(b)\alpha} = \partial\theta^a/\partial x^\alpha \ . \tag{4.11}$$

The triad $\{\underset{\sim}{e}_{(a)},\underset{\sim}{n}\}$ then spans the 3-space at each point. From (4.9), (4.10) and (4.11), or by simple modification of (2.11), it is immediate that

$$g^{\alpha\beta} = g^{ab}\, e^{\alpha}_{(a)}\, e^{\beta}_{(b)} + n^{\alpha} n^{\beta} \quad . \tag{4.12}$$

Using (4.8), it is easy to show

$$\rho^2 V_{|\alpha\beta} n^{\beta} = -\partial_{\alpha}\rho \quad . \tag{4.13}$$

The extrinsic curvature of the 2-space V = const., considered as imbedded in the 3-space t = const., is defined, from (2.4), by

$$\delta n^{\alpha}/\delta\theta^{b} = K^{a}_{b}\, e^{\alpha}_{(a)} \quad , \tag{4.14}$$

from which

$$K_{ab} = e^{\alpha}_{(a)} \delta n_{\alpha}/\delta\theta^{b} = e^{\alpha}_{(a)} n_{\alpha|\beta} e^{\beta}_{(b)} = \rho e^{\alpha}_{(a)} e^{\beta}_{(b)}\, V_{|\alpha\beta} \quad . \tag{4.15}$$

By virtue of (4.12), (4.13) and (4.15) we have

$$\begin{aligned} V_{|\alpha\beta} = \rho^{-1} K_{ab} e^{(a)}_{\alpha} e^{(b)}_{\beta} &- \rho^{-2}\partial_{c}\rho\,(e^{(c)}_{\alpha} n_{\beta} + e^{(c)}_{\beta} n_{\alpha}) \\ &- \rho^{-3}(\partial\rho/\partial V) n_{\alpha} n_{\beta} \quad . \end{aligned} \tag{4.16}$$

From (4.10), (4.11) and (4.14) we also obtain

$$K_{ab} = \frac{1}{2}\,\rho^{-1} \partial g_{ab}/\partial V \quad , \tag{4.17}$$

which leads to the related formula

$$\partial g^{\frac{1}{2}}/\partial V = g^{\frac{1}{2}}\,\rho K \quad , \tag{4.18}$$

where g is the 2×2 determinant of g_{ab} and $K = g^{ab} K_{ab}$ is twice the mean curvature. From (4.12) and (4.16) it follows that

$$V^{|\mu}_{\ |\mu} = \rho^{-1}K - \rho^{-3}\ \partial\rho/\partial V \quad . \tag{4.19}$$

The Einstein field equations

$$G_{AB} = -8\pi\gamma T_{AB} \ , \tag{4.20}$$

(where γ is Newton's constant of gravitation divided by c^2) can now be decomposed [84], yielding

$$\left.\begin{aligned} \frac{1}{2}\, g^{\alpha\beta}R_{\alpha\beta} &= 8\pi\gamma T^{o}_{o} \\ 0 &= 8\pi\gamma T_{\alpha o} \\ G_{\alpha\beta} = -8\pi\gamma T_{\alpha\beta} &- V^{-1}(V_{|\alpha\beta} - V^{|\mu}_{\ |\mu}\, g_{\alpha\beta}) \quad . \end{aligned}\right\} \tag{4.21}$$

The relativistic analogue of Poisson's equation is then given by

$$V^{|\mu}_{\ |\mu} = 4\pi\gamma V(T^{\alpha}_{\alpha} - T^{o}_{o}) \quad , \tag{4.22}$$

which, combined with (4.19), gives

$$\rho^{-2}\partial\rho/\partial V = K - 4\pi\gamma V\rho(T^{\alpha}_{\alpha} - T^{o}_{o}) \quad . \tag{4.23}$$

There are several other important relations which we require in what follows. However, their derivations are somewhat tedious, and we relegate them to Appendix D, merely listing the results here for later use:

$$V^{-1}g^{-\frac{1}{2}}\partial(g^{\frac{1}{2}}VK^{b}_{a})/\partial V = -\rho_{;a}^{\ ;b} - \frac{1}{2}\rho R\delta^{b}_{a} - 8\pi\gamma\rho(T_{\alpha\beta}e^{\alpha}_{(a)}e^{\beta}_{(b)} - \frac{1}{2}\, T^{A}_{A}g_{ab}), \tag{4.24}$$

$$\tfrac{1}{2}(K_{ab}K^{ab}-K^2-R) = -8\pi\gamma T_{\alpha\beta}n^\alpha n^\beta + \rho^{-1}V^{-1}K \quad , \tag{4.25}$$

$$\partial_a K - K^b_{a;b} = -8\pi\gamma T_{\alpha\beta}e^\alpha_{(a)}n^\beta + \rho^{-2}V^{-1}\partial_a\rho \quad , \tag{4.26}$$

$$\tfrac{1}{4}R_{ABCD}R^{ABCD} = G_{\alpha\beta}G^{\alpha\beta}+\rho^{-2}V^{-2}K_{ab}K^{ab}+2\rho^{-4}V^{-2}\rho_{;a}\rho^{;a}+\rho^{-6}V^{-2}(\partial\rho/\partial V)^2, \tag{4.27}$$

where $R = g^{ab}R_{ab}$. Equations (4.17), (4.23), and (4.24) form a complete system for determining the evalution of g_{ab}, ρ, K^b_a as functions of V. Equations (4.25) and (4.26) are involutive constraints, i.e., if they are satisfied on one surface V = const., they must be satisfied identically. The last equation gives the (invariant) square of the four-dimensional Riemann tensor.

§4.3 Static Massless Scalar Fields

We now consider a static scalar field Φ for zero-mass particles, with "scalar gradient" given by

$$\sigma_A = -(4\pi)^{\frac{1}{2}}\,\partial_A\Phi \quad . \tag{4.28}$$

Since Φ is static we have $\sigma_o = 0$. The scalar equation $\Box\Phi = 0$ can be written

$$(-\det g_{AB})^{-\frac{1}{2}}\partial_\alpha[(-\det g_{AB})^{\frac{1}{2}}g^{\alpha\beta}\partial_\beta\Phi] = 0 \quad . \tag{4.29}$$

In view of (4.7) and (4.9) this can also be written as

$$V^{-1}\partial(Vg^{\frac{1}{2}}\psi)/\partial V = -\partial_a(\rho g^{\frac{1}{2}}\Phi^{;a}) \quad , \tag{4.30}$$

with ψ defined by

$$\partial\Phi/\partial V = \rho\psi \quad . \tag{4.31}$$

For a massless scalar field the energy tensor is given by

$$T_{AB} = \partial_A\Phi\partial_B\Phi - \frac{1}{2} g_{AB}g^{CD}\partial_C\Phi\partial_D\Phi \quad , \tag{4.32}$$

or from (4.28) we can write

$$\left.\begin{aligned} 4\pi T^o_o &= -\frac{1}{2}\sigma^2 \quad , \\ 4\pi T^{o\alpha} &= 0 \ , \\ 4\pi T^{\alpha\beta} &= \sigma^\alpha\sigma^\beta - \frac{1}{2} g^{\alpha\beta}\sigma^2 \ , \end{aligned}\right\} \tag{4.33}$$

with

$$\left.\begin{aligned} \sigma_\alpha &= -(4\pi)^{\frac{1}{2}}(\psi n_\alpha + e_\alpha^{(a)}\partial_a\Phi) \quad , \\ \sigma^2 \equiv \sigma_\alpha\sigma^\alpha &= 4\pi(\psi^2 + \Phi_{;a}\Phi^{;a}) \quad . \end{aligned}\right\} \tag{4.34}$$

If we now substitute (4.33) into the basic equations (4.23)-(4.26) of the previous section, we are led to the following complete first-order system for determining the V-dependence of g_{ab}, Φ, ψ, ρ, and K^b_a:

<u>Geometrical equation</u>

$$\partial g_{ab}/\partial V = 2\rho K_{ab} \quad , \tag{4.17}$$

<u>Static Scalar equations</u>

$$\partial\Phi/\partial V = \rho\psi \quad , \tag{4.31}$$

$$\partial(Vg^{\frac{1}{2}}\psi)/\partial V = -V\partial_a(\rho g^{\frac{1}{2}}\Phi^{;a}) \quad , \tag{4.30}$$

Gravitational equations

$$\rho^{-2}\partial\rho/\partial V = K \quad , \tag{4.35}$$

$$V^{-1}g^{-\frac{1}{2}}\partial(g^{\frac{1}{2}}VK_a^b)/\partial V = -\rho_{;a}{}^{;b} - \frac{1}{2}\rho R\delta_a^b - 8\pi\gamma\rho\Phi_{;a}\Phi^{;b} \quad , \tag{4.36}$$

Involutive constraints

$$\frac{1}{2}(K_{ab}K^{ab}-K^2-R) = -4\pi\gamma(\psi^2-\Phi_{;a}\Phi^{;a}) + \rho^{-1}V^{-1}K \quad , \tag{4.37}$$

$$\partial_a K - K_{a;b}^b = -8\pi\gamma\psi\Phi_{;a} + \rho^{-2}V^{-1}\rho_{;a} \quad . \tag{4.38}$$

The following result, which we will need later, is obtained by contracting (4.36) and eliminating R by means of (4.37):

$$V\partial(V^{-1}K)/\partial V = -\rho_{;a}{}^{;a} - \frac{1}{2}\rho K^2 - \rho\Lambda_{ab}\Lambda^{ab} - 8\pi\gamma\rho\psi^2 \quad , \tag{4.39}$$

where

$$\Lambda_{ab} = K_{ab} - \frac{1}{2}g_{ab}K \tag{4.40}$$

is a measure of deviation from spherical symmetry.

Finally, we combine (4.27) and (4.35) to obtain

$$\frac{1}{4}R_{ABCD}R^{ABCD} = G_{\alpha\beta}G^{\alpha\beta}+\rho^{-2}V^{-2}K_{ab}K^{ab}+2\rho^{-4}V^{-2}\rho_{;a}\rho^{;a}+\rho^{-2}V^{-2}K^2 \quad . \tag{4.41}$$

Although an exact expression for the quantity $G_{\alpha\beta}G^{\alpha\beta}$ could be evaluated from (4.12), (4.16), (4.21), (4.22), (4.33), and (4.34), it is sufficient

to note here that all terms of this expression will be positive, with one such term being

$$48\pi^2\gamma^2(\psi^2+\Phi_{;a}\Phi^{;a})^2 \quad . \tag{4.42}$$

§4.4 Statement of Theorem

In a static space-time, let $\sum$ be any spatial hypersurface t = const., maximally extended consistent with $\xi_A\xi^A < 0$. We consider the class of static massless scalar fields such that the following conditions are satisfied on $\sum$:

(i) $\sum$ is a "scalar-vacuum" space (i.e., free of matter and sources of scalar fields).

(ii) $\sum$ is regular, non-compact and "asympotically Euclidean". That is, there exist coordinates x^α in terms of which the metric (4.7) has the asymptotic form

$$\left.\begin{aligned} &g_{\alpha\beta} = \delta_{\alpha\beta} + 0(r^{-1}),\ \partial_\gamma g_{\alpha\beta} = 0(r^{-2}) \ , \\ &V = 1 - (m/r) + \eta \ , \ m = \text{const.}, \\ &\eta = 0(r^{-2}),\ \partial_\alpha\eta = 0(r^{-3}),\ \partial_\alpha\partial_\beta\eta = 0(r^{-4}) \end{aligned}\right\} (r \to \infty) \tag{4.43}$$

where $r = (\delta_{\alpha\beta}x^\alpha x^\beta)^{\frac{1}{2}}$.

(iii) The asymptotic form of the static scalar field is

$$\left.\begin{aligned} &\Phi = (k/r) + \zeta \ , \quad k = \text{const.}, \\ &\zeta = 0(r^{-2}), \qquad \partial_\alpha\zeta = 0(r^{-3}) \ . \end{aligned}\right\} (r \to \infty) \tag{4.44}$$

(iv) The equipotential surfaces $V = \text{const.} > 0$, $t = \text{const.}$, are a regular family of simply-connected, closed 2-spaces.

(v) If the greatest lower bound of V on $\sum$ is zero, then the geometry of the equipotential surface $V = \varepsilon$ approaches a limit as $\varepsilon \to 0_+$, corresponding to a closed regular horizon of non-infinite area.

(vi) The invariant $R_{ABCD}R^{ABCD}$ is bounded on $\sum$.

<u>Theorem</u>: There is no non-trivial static space-time which satisfies conditions (i) - (vi).

(Here we assume $\gamma > 0$. The case of zero coupling is discussed in §4.6.)

The proof of the theorem is presented in §4.5. There is one trivial case, however, which can be quickly disposed of here. Suppose that V has a positive lower bound. Then the maximally extended 3-space $\sum$ is complete. Using (4.29) and the boundary condition (4.44), Green's theorem implies $\Phi \equiv 0$. Furthermore (4.22) reduces, as a result of comparing (4.23) and (4.35), to Laplace's equation $V^{|\mu}{}_{|\mu} = 0$, which, together with the boundary conditions (4.43), leads to $V \equiv 1$. This means space-time is flat, and the theorem is established.

We may assume henceforth that V comes arbitrarily close to zero on $\sum$. The equipotential surface $V = 0_+$ then forms an inner boundary of $\sum$. Suppose that V has zero gradient at some interior point P of $\sum$. Since V is harmonic $(V^{|\mu}{}_{|\mu} = 0)$ on $\sum$, P would have to be a point of bifurcation of the equipotential surfaces [85], which

contradicts (iv). Therefore, by (4.8), ρ^{-1} vanishes nowhere in the domain of interest.

We conclude this section by recording the exterior and interior boundary conditions in a form convenient for later application. For the asymptotic forms (4.43) and (4.44), we find from (4.8), (4.18), and (4.31)

$$\left.\begin{array}{l} r \to \infty, \quad \rho/r^2 \to m^{-1}, \quad rk \to 2, \\ r\Phi \to k, \quad r^2\psi \to -k, \quad \text{as} \quad V \to 1 \ . \end{array}\right\} \qquad (4.45)$$

According to (vi) and (4.41), the regularity of the manifold at the inner boundary $V = 0_+$ requires that

$$K_{ab} = O(\rho V), \quad \rho_{;a} = O(\rho^2 V) \quad \text{as} \quad V \to 0_+ \ . \qquad (4.46)$$

It follows that ρ^{-1} is constant on the event horizon:

$$\rho^{-1}(0,\theta^1,\theta^2) = 1/\rho_o = \text{const.} \qquad (4.47)$$

In addition, since the curvature scalar $R_{ABCD}R^{ABCD}$ is bounded everywhere on Σ by (vi), it follows that the expression (4.42) cannot become infinite anywhere on Σ. Hence we have the result that ψ and $\Phi_{;a}$ are bounded functions throughout Σ.

§4.5 Proof of Theorem

In this section we employ mainly the scalar equations, (4.30) and (4.31), to derive integral relations which enable us to show that the trivial solution $\Phi \equiv 0$ is the only one compatible with conditions (i) - (vi).

Let $F(V,\Phi)$, $G(V,\Phi)$ be (for the moment, arbitrary) differentiable functions. From (4.30), (4.31), (4.35), and (4.18) we easily obtain the identity

$$g^{-\frac{1}{2}}(\partial/\partial V)[g^{\frac{1}{2}}\{VF(V,\Phi)\psi+\rho^{-1}G(V,\Phi)\}] = A(V,\Phi)\rho(\psi^2+\Phi_{;a}\Phi^{;a}) + B(V,\Phi)\psi$$
$$+ \rho^{-1}\partial G/\partial V - Vg^{-\frac{1}{2}}(F\rho g^{\frac{1}{2}}\Phi^{;a})_{;a} \ , \tag{4.48}$$

where

$$A \equiv V\partial F/\partial\Phi, \quad B \equiv V\partial F/\partial V+\partial G/\partial\Phi \ . \tag{4.49}$$

In order to obtain integral conservation laws from (4.48) and (4.49), we require that

$$A = B = \partial G/\partial V = 0 \ . \tag{4.50}$$

The general solution of this linear system of differential equations for F, G is a linear combination of the two particular solutions

$$\left.\begin{array}{l} F = 1, \quad G = 1 \ , \\ F = \ell n\ V, \quad G = -\Phi \ . \end{array}\right\} \tag{4.51}$$

Taking these values in turn we integrate (4.48) over $\sum$, i.e., we form $\int\int_{\Sigma}\int$ (4.48) $g^{\frac{1}{2}}dVd\theta^1 d\theta^2$, noting that the integral of the last term, being a 2-divergence, vanishes when taken over any closed 2-space V = const. The results express the equality of the surface integrals of the expression in square brackets above over any two equipotential surfaces V = const.:

$$\int (V\psi+\rho^{-1})dS = C_1 \tag{4.52}$$

$$\int [(V\ \ell n\ V)\psi-\rho^{-1}\Phi]dS = C_2 \ , \tag{4.53}$$

where we have defined the element of area by $dS = g^{\frac{1}{2}}d\theta^1 d\theta^2$.

As an immediate consequence of (4.18) and (4.35) we have

$$\partial(\rho^{-1}g^{\frac{1}{2}})/\partial V = 0 \ , \tag{4.54}$$

hence if we form $\iiint$ (4.54) $dVd\theta^1 d\theta^2$, we obtain

$$\int \rho^{-1}dS = C_3 \ . \tag{4.55}$$

Comparison of (4.52) and (4.55) show that

$$\int V\psi dS = C_1 - C_3 = C_4 \ . \tag{4.56}$$

We can now evaluate the constants C_2, C_3 and C_4 by integrating (4.53), (4.55) and (4.56) over the upper boundary $V = 1$, with the help of the boundary conditions (4.45). We thus find, as integral conditions on the lower boundary $V = 0_+$:

$$\int_{V=0_+} \rho^{-1}dS = 4\pi m \ , \tag{4.57}$$

$$\int_{V=0_+} V\psi dS = -4\pi k \ , \tag{4.58}$$

$$\int_{V=0_+} [(V\ \ell n\ V)\psi-\rho^{-1}\Phi]dS = 0 \ . \tag{4.59}$$

In view of (4.47) we can write (4.57) as

$$S_o/\rho_o = 4\pi m \quad , \tag{4.60}$$

where S_o is the area of the horizon $V = 0_+$. Since S_o is non-infinite by (v), (4.60) implies that ρ_o is also non-infinite.

We now consider the relation (4.58). Since ψ and the surface area are bounded on the horizon $V = 0_+$, it follows that $k \equiv 0$, i.e.,

$$\int_{V=0_+} V\psi dS = 0 \quad . \tag{4.61}$$

In addition, the boundedness of $\rho\psi = \partial\Phi/\partial V$ and $\Phi_{;a}$ throughout $\sum$ guarantees that Φ itself is bounded on the horizon. This fact, together with (4.61) leads to

$$\int_{V=0_+} V\Phi\psi dS = 0 \quad . \tag{4.62}$$

We now return to the identity (4.48)-(4.49) and this time require

$$A = V, \quad B = \partial G/\partial V = 0 \quad . \tag{4.63}$$

The resulting linear differential equations have the particular solution

$$\begin{aligned} F &= \Phi + \ln V \\ G &= -\Phi \quad . \end{aligned} \tag{4.64}$$

With F, G given by (4.64), we thus have the identity

$$g^{-\frac{1}{2}}(\partial/\partial V)[g^{\frac{1}{2}}(VF\psi+\rho^{-1}G)] = \rho V(\psi^2+\Phi_{;a}\Phi^{;a}) - Vg^{-\frac{1}{2}}(F\rho g^{\frac{1}{2}}\Phi^{;a})_{;a} \quad . \tag{4.65}$$

Integrating over $\sum$ (again the last term doesn't contribute) we deduce

the inequality

$$\int_{V=1} (VF\psi+\rho^{-1}G)dS \geq \int_{V=0_+} (VF\psi+\rho^{-1}G)dS \quad . \tag{4.66}$$

From (4.65) it is clear that equality in (4.66) holds if and only if

$$\left.\begin{aligned} \psi &\equiv 0 \\ \Phi_{;a} &\equiv 0 \end{aligned}\right\} \tag{4.67}$$

<u>everywhere</u> on $\sum$. Now both surface integrals in (4.66) can actually be evaluated using (4.64). The left side yields the value zero when integrated with the help of (4.45), as does the right side, in view of (4.59) and (4.62). We conclude that (4.67) must hold, hence Φ = const. throughout $\sum$. The fact that Φ vanishes on the outer boundary $V = 1$ ensures that $\Phi \equiv 0$ and this completes the proof. It is worth noting that the theorem holds even for a regular point or line horizon $(S_o = 0)$.

§4.6 <u>Zero Coupling</u>

We consider in this section the case where the gravitational coupling of the scalar energy density is neglected. Our problem is to obtain solutions of the vacuum equation $G_{AB} = 0$ and the scalar equation $\Box\Phi = 0$, which satisfy conditions (i) - (vi) of §4.4. It is already known [86] that the only vacuum space-times compatible with (i) - (vi) are the Schwarzschild solutions with $m \geq 0$. The problem thus reduces to finding well-behaved static scalar fields defined on the Schwarzschild background:

$$\left.\begin{aligned} g_{\alpha\beta}dx^\alpha dx^\beta &= (1-2m/r)^{-1}dr^2 + r^2(d\theta^2+\sin^2\theta d\phi^2) , \\ V &= (1-2m/r)^{\frac{1}{2}} . \end{aligned}\right\} \qquad (4.68)$$

The static scalar equation, $\Box\Phi = 0$, reduces to

$$\frac{V^2}{r^2}\frac{\partial}{\partial r}\left(V^2r^2\frac{\partial\Phi}{\partial r}\right) + \frac{V^2}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{V^2}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} = 0 . \qquad (4.69)$$

Separable solutions which are regular on the axis have the form

$$\Phi = R(r)P_n^M(\cos\theta)e^{iM\phi} , \qquad (4.70)$$

where R satisfies

$$(1-x^2)\frac{d^2R}{dx^2} - 2x\frac{dR}{dx} + n(n+1)R = 0 , \qquad (4.71)$$

$$x = \frac{r}{m} - 1 . \qquad (4.72)$$

For general n, (4.71) has the linearly independent solutions

$$R = C_1\mathfrak{P}_n(x), \qquad R = C_2\mathfrak{Q}_n(x) . \qquad (4.73)$$

Of these solutions, the latter are unacceptable for all n, since $\mathfrak{Q}_n(x)$ is singular at $x = 1$ (i.e., at the event horizon $r = 2m$). On the other hand, the former solutions have the wrong behaviour at infinity, except when $n = 0$. Hence the only well-behaved solution satisfying (i) - (vi) occurs for $n = M = 0$ and is given by

$$\Phi = C_1 = 0 \quad ,$$

from the boundary condition (4.44). We conclude that the trivial solution $\Phi = 0$ is the only static scalar field on a Schwarzschild background which is well-vehaved for $2m \leq r < \infty$.

§4.7 Singular Point Horizons

It is clear from the previous theorem that a non-trivial static scalar-vacuum field must have a singular event horizon (be it a point or otherwise), and thus Penney's search [87] for an asymmetric solution with a regular event horizon is unnecessary. Our theorem also generalizes the work of JNW [88] who show that all spherically symmetric solutions always have singular horizons.

In the case of spherical symmetry, equations (4.30) and (4.39) become simply

$$\partial(Vg^{\frac{1}{2}}\psi)/\partial V = 0 \quad , \tag{4.74}$$

$$V\partial(V^{-1}K)/\partial V = -\frac{1}{2}\rho K^2 - 8\pi\gamma\rho\psi^2 \quad . \tag{4.75}$$

Now (4.74), (4.54) and (4.31) can be solved explicitly with the help of the asymptotic forms (4.45) to give

$$\Phi = -km^{-1} \ln V \quad , \tag{4.76}$$

which is a function of the two parameters k and m. Consideration of (4.35), (4.75) and (4.76) enables us to solve for ρ and K as functions of V, from which it can be determined that $\rho \to 0$ as $V \to 0_+$. From (4.57) it then follows that the area of the inner boundary $V = 0_+$ is zero, i.e., the horizon is a point (the non-regularity of the event horizon means that (4.47) is no longer true, but the integral condition (4.57) still holds). This solution is precisely the JNW solution [89], which has a singular point horizon regardless of how small the coupling constant becomes.

The fact that the horizon is always a point in the case of spherical symmetry leads us to ask whether the same is true in general. Penney's example [90], although not regular as he had thought [91], serves to show that there are asymmetric solutions whose horizons, while singular, are not point-like. His axially symmetric solution has the line element

$$ds^2 = e^{2\nu} \left[\left(\frac{R}{R-2m}\right) dR^2 + R^2 d\theta^2\right] + R^2 \sin^2\theta d\phi^2 - \left(\frac{R-2m}{R}\right) dt^2 \ , \tag{4.77}$$

with

$$\nu = -2\pi\gamma a^2 \ \frac{R(R-2m)\sin^2\theta}{[R(R-2m)+m^2\cos^2\theta]^2} \ . \tag{4.78}$$

It is easy to check that the horizon $(R = 2m)$ is not point-like. This example, along with (4.57) suffices to show that the function ρ need not vanish everywhere on the horizon $V = 0_+$. We can, however, show

for a general scalar solution with event horizon of bounded area, that ρ always vanishes at least locally there. From (4.35) and (4.54) it is easily shown that

$$\frac{d}{dV} \int \rho^{-2} \, dS = -\int \rho^{-1} K \, dS \quad . \tag{4.79}$$

From (4.39) and (4.54) we have

$$\begin{aligned} \partial(\rho^{-1} g^{\frac{1}{2}} K)/\partial V = \rho^{-1} V^{-1} g^{\frac{1}{2}} K - 8\pi\gamma g^{\frac{1}{2}} \psi^2 - \frac{1}{2} g^{\frac{1}{2}} K^2 \\ - g^{\frac{1}{2}} [(\ell n \, \rho)_{;a}{}^{;a} + \rho^{-2} \rho_{;a} \rho^{;a} + \Lambda_{ab} \Lambda^{ab}] \quad . \end{aligned} \tag{4.80}$$

If we now form $\iiint$ (4.80) $dV d\theta^1 d\theta^2$ and use the Schwarz inequality on the second term of the right side,

$$\int \psi^2 dS \geq \frac{1}{S} [\int \psi dS]^2 \quad ,$$

it is straightforward to arrive at the inequality

$$\frac{d}{dV} \int K \rho^{-1} \, dS \leq \frac{1}{V} \int K\rho^{-1} \, dS - \frac{8\pi\gamma}{S} [\int \psi dS]^2 \quad , \tag{4.81}$$

which, in view of (4.56) can be written

$$\frac{d}{dV} \int K \rho^{-1} \, dS \leq \frac{1}{V} \int K \rho^{-1} \, dS - \frac{8\pi\gamma C_4^2}{V^2 S} \quad . \tag{4.82}$$

If we now let

$$X(V) = \int \rho^{-2}\, dS \quad , \tag{4.83}$$

then (4.79) and (4.82) combine to give

$$\frac{d}{dV}\left[\frac{1}{V}\frac{dX}{dV}\right] \geq \frac{C}{V^3 S} \quad , \qquad C = \text{const.} > 0 \quad . \tag{4.84}$$

Since $S = S(V)$ is a continuous function, infinite only on the outer boundary $V = 1$, then for any $V_1 < 1$, $S(V)$ must attain a maximum S_M somewhere in the interval from $V = 0_+$ to $V = V_1$. In this interval we thus have $1/S \geq 1/S_M$, hence

$$\frac{d}{dV}\left[\frac{1}{V}\frac{dX}{dV}\right] \geq \frac{C'}{V^3} \quad , \qquad C' = C/S_M > 0 \quad . \tag{4.85}$$

Integrating (4.85) twice we arrive at the inequality

$$\left.\begin{aligned} &X(V) \geq -C_1 \ln V + C_2 V^2 + C_3 \ , \\ &C_1 = C'/2 > 0, \quad C_2 \text{ and } C_3 \ \text{ arbitrary} \ , \end{aligned}\right\} \tag{4.86}$$

from which

$$X(0) = \int_{V=0_+} \rho^{-2}\, dS = \infty \quad . \tag{4.87}$$

The integral condition (4.87) along with (4.57) shows that ρ becomes zero at least locally on the inner boundary $V = 0_+$ (this reaffirms that for spherical symmetry the horizon must be a point). It is also clear from (4.57) that the region of the horizon on which ρ vanishes must be of zero area. In other words the gravitational flux $\rho^{-1} = |\nabla V|$

becomes infinite somewhere on the horizon because some finite-area flux tube shrinks to a point or line there.

In the case of Penney's example, comparison of (4.7) and (4.9) with (4.77) shows

$$V = \left(\frac{R-2m}{R}\right)^{\frac{1}{2}} \quad , \tag{4.88}$$

$$\rho^2 dV^2 = e^{2\nu} \left(\frac{R}{R-2m}\right) dR^2 \quad . \tag{4.89}$$

From (4.88) and (4.89) we immediately deduce that

$$\rho = e^{\nu} R^2/m \quad . \tag{4.90}$$

On the horizon $R = 2m$, consideration of (4.78) shows that ρ vanishes in the equatorial plane $\theta = \pi/2$.

We can summarize our results for static scalar fields as follows:

(1) Every massless scalar field which is gravitationally coupled, static and asymptotically flat, becomes singular at a simply-connected event horizon.

(2) In the case of spherical symmetry, the singular horizons are points.

(3) If we allow asymmetric solutions, it is possible to find examples for which the horizon is not point-like.

(4) Assuming the horizon has bounded surface area, the gravitational flux ρ^{-1} always becomes singular somewhere on the horizon, due to the shrinking of some finite-area flux tube to a point or line.

APPENDIX A

2-Dimensional Kinetic Theory of a Simple Relativistic Gas

Consider a 2-dimensional distribution (for example a spherical shell) of a relativistic ideal gas. Let $n(p)dp$ be the number of particles with momenta $(p,p+dp)$ per unit area, as measured in the rest frame of the gas.

The baryon density of the gas is given by

$$\sigma_o = \int_o^\infty n(p) m_A dp \quad , \tag{A.1}$$

where m_A is the atomic mass of a constituent particle. The total energy of an atom with velocity u is

$$E = m_A (1-u^2)^{-\frac{1}{2}} \quad , \tag{A.2}$$

hence the total surface energy density is

$$\sigma = \int_o^\infty n(p) E \, dp = \int_o^\infty n(p) m_A (1-u^2)^{-\frac{1}{2}} dp \quad . \tag{A.3}$$

The pressure exerted by the gas is the normal component of the mean rate of momentum transfer across a line segment of unit length, and is given by

$$P = \frac{1}{2} \int_o^\infty n(p) p u \, dp \quad , \tag{A.4}$$

where the factor $\frac{1}{2}$ arises from averaging $\cos^2\theta$ over all directions in a plane and the momentum p of an atom is simply

$$p = m_A u(1-u^2)^{-\frac{1}{2}} \quad . \tag{A.5}$$

We assume that pressure is a monotone increasing function of surface energy σ, and discuss the special limiting cases of the equation of state, namely the non-relativistic limit $(P << \sigma_o)$, and the ultra-relativistic limit $(P >> \sigma_o)$.

In the non-relativistic case we have $u << 1$, hence $E \approx m_A(1 + u^2/2)$ and $p \approx m_A u$. The integrand in (A.3) then vanishes except for very small p, and we have

$$\sigma \approx \sigma_o + \frac{1}{2}\int_o^\infty n(p) m_A u^2 dp \approx \sigma_o + P \quad . \tag{A.6}$$

Substituting (A.6) into the isentropic condition

$$d(\sigma/\sigma_o) + P\, d(1/\sigma_o) = 0 \quad , \tag{A.7}$$

it is found that

$$P \propto \sigma_o^2 \quad . \tag{A.8}$$

The dependence of P on σ, since $P << \sigma_o$, is immediate from (A.6) and (A.8), hence

$$P \propto \sigma^2 \quad . \tag{A.9}$$

For the ultra-relativistic case, $u \to 1$, thus $E \simeq p$, from (A.2) and (A.5). This time the main contribution to the integrand in (A.3) is from very large p. It is then immediate from (A.3) and (A.4) that

$$\sigma \approx \int_{o}^{\infty} n(p)p \, dp \approx 2P \quad , \tag{A.10}$$

i.e., $P \, \alpha \, \sigma$, the fully relativistic analogue of (A.9). Putting (A.10) into (A.7) leads to

$$P \, \alpha \, \sigma_o^{3/2} \quad , \tag{A.11}$$

corresponding to (A.8). Equations (A.8) and (A.11) are important, since they show that, for an ultra-relativistic gas exerting pressure, the adiabatic exponent $\gamma = 3/2$, and that otherwise $3/2 < \gamma \leq 2$.

We conclude by showing that $\sigma - 2P < \sigma_o$ for a simple 2-dimensional relativistic gas, interacting only by collisions. Since $p = m_A u(1-u^2)^{-\frac{1}{2}}$, it follows from (A.1) , (A.3) and (A.4) that

$$\sigma - 2P = \int_{o}^{\infty} n(p)m_A(1-u^2)^{\frac{1}{2}} \, dp < \sigma_o \quad .$$

APPENDIX B

Electromagnetic Field Equations on a Schwarzschild Background

Consider the Schwarzschild background metric in the form

$$\left.\begin{aligned} ds^2 &= \alpha dx dy + r^2 \Omega^2 \quad , \\ \alpha &= 1 - \frac{1}{r} \quad , \end{aligned}\right\} \tag{B.1}$$

where

$$\left.\begin{aligned} x &= (r-1) + \ell n(r-1) - t \quad , \\ y &= (r-1) + \ell n(r-1) + t \quad . \end{aligned}\right\} \tag{B.2}$$

With $x^\mu = (x,y,\theta,\phi)$, we can then write

$$[g_{\mu\nu}] = \begin{bmatrix} 0 & \alpha/2 & 0 & 0 \\ \alpha/2 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{bmatrix} , \quad [g^{\mu\nu}] = [g_{\mu\nu}]^{-1} . \tag{B.3}$$

We are concerned with the electromagnetic field equations

$$F^{\mu\nu}{}_{|\nu} = \frac{1}{\sqrt{-G}} (\sqrt{-G}\, F^{\mu\nu})_{,\nu} = 0 \tag{B.4}$$

on the background (B.1), where

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \quad . \tag{B.5}$$

Assuming the field is axi-symmetric (independent of $x^3 = \phi$) and evaluating $G = \det g_{\mu\nu} = -\frac{\alpha^2}{4} r^4 \sin^2\theta$ from (B.3), we can write (B.4) as

$$\frac{2}{\alpha r^2 \sin\theta} \left(\frac{\alpha}{2} r^2 \sin\theta \, F^{\mu j}\right)_{,j} = 0 \qquad (j = 0,1,2) \quad ,$$

or equivalently,

$$\left(\frac{\alpha}{2} r^2 \sin\theta \, g^{\mu\rho} \, g^{j\sigma} \, F_{\rho\sigma}\right)_{,j} = 0 \quad . \tag{B.6}$$

Putting $\mu = 3$ in (B.6) we obtain

$$\left(\frac{\alpha}{2\sin\theta} g^{01} F_{31}\right)_{,0} + \left(\frac{\alpha}{2\sin\theta} g^{10} F_{30}\right)_{,1} + \left(\frac{\alpha}{2\sin\theta} g^{22} F_{32}\right)_{,2} = 0 \quad ,$$

which from (B.3) and (B.5) simplifies to

$$\frac{2}{\sin\theta} A_{3,01} + \frac{\alpha}{2r^2} \left(\frac{1}{\sin\theta} A_{3,2}\right)_{,2} = 0 \quad . \tag{B.7}$$

In a similar fashion, (B.6) with $\mu = 2$ becomes

$$A_{1,02} + A_{0,12} - 2A_{2,01} = 0 \quad . \tag{B.8}$$

Now we will use the fact that the field is gauge invariant, i.e., it is unaltered by a transformation of the type

$$A_\mu = A_\mu' + \Phi_{,\mu} \quad .$$

Consequently

$$A_{1,0} + A_{0,1} = A'_{1,0} + A'_{0,1} + 2\Phi_{,01} \quad .$$

Now the wave equation $\Phi_{,01} = h(x,y,\theta)$ can always be solved; in particular we can choose our Φ such that

$$\Phi_{,01} = -\frac{1}{2}(A'_{1,0} + A'_{0,1}) \quad ,$$

so that

$$A_{1,0} + A_{0,1} = 0 \quad , \tag{B.9}$$

can always be made to hold. In view of (B.9), then, (B.8) becomes simply

$$A_{2,01} = 0 \quad ,$$

from which

$$A_2 = f(x,\theta) + g(y,\theta), \qquad f \text{ and } g \text{ arbitrary.} \tag{B.10}$$

Finally we put $\mu = 1$ and $\mu = 0$ in (B.6) and obtain similar relations, namely

$$\left(\frac{2r^2}{\alpha} F_{01}\right)_{,0} + \frac{1}{\sin\theta} (\sin\theta\, F_{02})_{,2} = 0 \quad , \tag{B.11}$$

$$\left(\frac{2r^2}{\alpha} F_{10}\right)_{,1} + \frac{1}{\sin\theta} (\sin\theta F_{12})_{,2} = 0 \quad . \tag{B.12}$$

It is easily shown from (B.2) that

$$\left(\frac{2r^2}{\alpha}\right)_{,0} = \left(\frac{2r^2}{\alpha}\right)_{,1} = \frac{2r-3}{\alpha} \quad , \tag{B.13}$$

and hence (B.11) and (B.12) become

$$\frac{2r-3}{\alpha} F_{01} + \frac{2r^2}{\alpha} F_{01,0} + F_{02,2} + \cot\theta \, F_{02} = 0 \quad , \tag{B.14}$$

$$\frac{2r-3}{\alpha} F_{10} + \frac{2r^2}{\alpha} F_{10,1} + F_{12,2} + \cot\theta \, F_{12} = 0 \quad . \tag{B.15}$$

Now from (B.5) and the gauge condition (B.9) we have

$$F_{01} = -2A_{0,1} \; , \qquad F_{10} = -2A_{1,0} \quad . \tag{B.16}$$

Finally, using (B.5), (B.13) and (B.16), we can put (B.14) and (B.15) in the form

$$4\left(\frac{r^2}{\alpha} A_{0,1}\right)_{,0} + \frac{1}{\sin\theta} \left(\sin\theta \, A_{0,2}\right)_{,2} = \frac{1}{\sin\theta}\left(\sin\theta \, A_{2,0}\right)_{,2} , \tag{B.17}$$

$$4\left(\frac{r^2}{\alpha} A_{1,0}\right)_{,1} + \frac{1}{\sin\theta}\left(\sin\theta \, A_{1,2}\right)_{,2} = \frac{1}{\sin\theta}\left(\sin\theta \, A_{2,1}\right)_{,2} \quad . \tag{B.18}$$

The four key equations to be solved for A_ν are therefore (B.7), (B.10), (B.17) and (B.18).

It is worth noting that the left sides of (B.17) and (B.18) involve only A_0 and A_1, respectively. This makes it particularly easy to see that $A_0 = A_1 = A_2 = 0$ trivially satisfies (B.10), (B.17) and (B.18).

APPENDIX C

Jump Conditions for the Electromagnetic Field

We are concerned here with finding the jump conditions for the electromagnetic field across an arbitrary 3-surface $\sum$. Suppose we choose a 3-dimensional "cylinder" R_3 of "length" 2ε and interior R_4, which cuts $\sum$ as shown in Figure 5. Applying Stokes' Theorem to an arbitrary vector quantity V^μ, we have

$$\int_{R_3} V_\mu \, d\tau^\mu = \int_{R_4} V^\mu{}_{|\mu} \, d\tau \quad , \tag{C.1}$$

where $d\tau^\mu$ is choosen normal to the 3-surface R_3.

We now let $V^\mu = F^{\mu\nu} \eta_\nu$, where $F^{\mu\nu}$ is the electromagnetic field tensor and η_ν is arbitrary. Then (C.1) becomes

$$\int_{R_3} F_{\mu\nu}\eta^\nu \, d\tau^\mu = \int_{R_4} F^{\mu\nu}{}_{|\mu}\eta_\nu \, d\tau + \int_{R_4} F^{\mu\nu}\eta_{\nu|\mu} \, d\tau \quad . \tag{C.2}$$

The first term on the right side of (C.2) vanishes.by virtue of the electromagnetic field equations (3.15). In the limit as $\varepsilon \to 0$, the volume R_4 becomes sufficiently small that the right side disappears entirely, while only the ends of the cylinder R_3 contribute to the left side. Therefore, since η^ν is arbitrary; (C.2) becomes

$$F_{\mu\nu} \, d\tau^\mu|^+ - F_{\mu\nu} \, d\tau^\mu|^- \equiv [F_{\mu\nu} \, d\tau^\mu]_\sum = 0 \quad ,$$

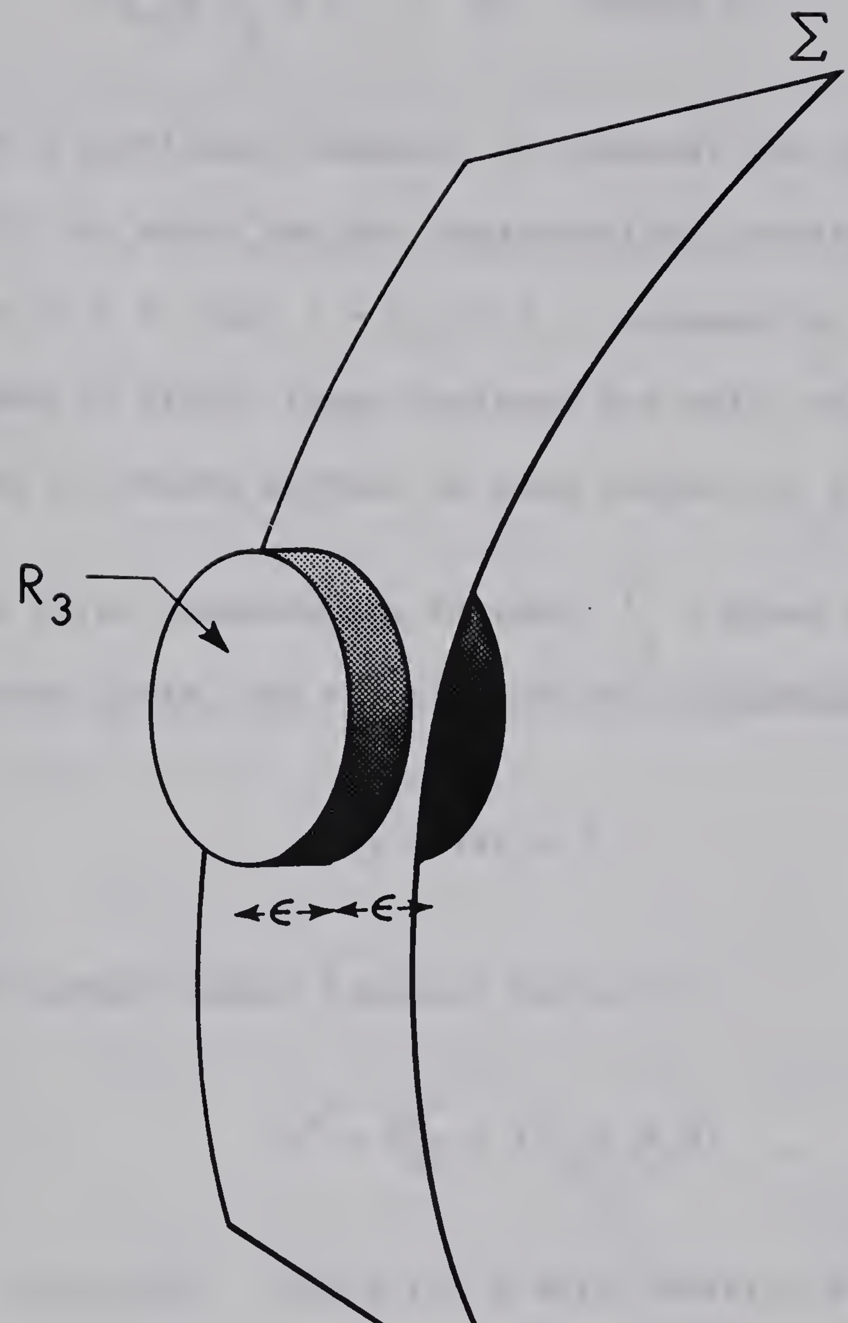


FIGURE 5

The 3-surface Σ being cut by a 3-cylinder R_3 of length 2ϵ. One dimension is suppressed in the diagram.

where $d\tau^{\mu}$ is normal to $\sum$. The required jump conditions are thus given by

$$[F_{\nu\beta}N^{\nu}]_{\sum} = 0 \quad , \quad N^{\nu} \text{ normal to } \sum . \tag{C.3}$$

As a particular example, we consider the idealized collapse model of §3.3, in which the two characteristic surfaces of interest are given by $y = 0$ and $x = x_o >> 1$. Because the collapse occurs with the speed of light, these surfaces are null, which means that a vector normal to either surface is also tangent to that surface.

We first consider the surface $\sum_1$, given by $y = 0$. For the static interior field, the equation of the collapsing shell is

$$y = r+t = 0 \quad ,$$

from which a normal (hence tangent) vector is

$$N^{\mu} = \frac{d\overline{x}^{\mu}}{dr} = (-1,1,0,0) \quad , \tag{C.4}$$

where $\overline{x}^{\mu} = (t,r,\theta,\phi)$. Using the static interior solution

$$\overline{A}_o = \overline{A}_1 = \overline{A}_2 = 0 \quad , \quad \overline{A}_3 = \frac{\mu \sin^2\theta}{r} \quad ,$$

found in §3.3, together with (C.4) and

$$F_{\mu\nu}\Big|^{+}_{-} = \overline{A}_{\mu,\nu} - \overline{A}_{\nu,\mu}\Big|^{+}_{-} \quad , \tag{C.5}$$

we obtain

$$\left.\begin{array}{ll} \underline{\mu=0,1,2}: & F_{\mu\nu} N^{\nu}\Big|_{\bar{\Sigma}_1} = 0 \quad , \\ \underline{\mu=3}: & F_{\mu\nu} N^{\nu}\Big|_{\bar{\Sigma}_1} = -\dfrac{\mu \sin^2\theta}{r^2} \end{array}\right\} \tag{C.6}$$

Turning to the exterior field, the equation of the shell is

$$y = (r-1) + \ell n(r-1) + t = 0 \quad ,$$

and hence

$$N^{\mu} = \frac{d\bar{x}^{\mu}}{dr} = (-\frac{1}{\alpha},1,0,0) \quad , \quad \alpha = \frac{(r-1)}{r} \ . \tag{C.7}$$

Using (C.3) , (C.5) , (C.6) and (C.7) we easily obtain the jump conditions across $y = 0$, namely:

$$\left.\begin{array}{ll} \underline{\mu=0,1}: & (\bar{A}_{o,1} - \bar{A}_{1,o})\Big|^{+}_{\Sigma_1} = 0 \\ \underline{\mu=2}: & (\bar{A}_{o,2} - \bar{A}_{2,o})(-\frac{1}{\alpha})\Big|^{+}_{\Sigma_1} + (\bar{A}_{1,2} - \bar{A}_{2,1})\Big|^{+}_{\Sigma_1} = 0 \\ \underline{\mu=3}: & (\bar{A}_{3,1} - \bar{A}_{3,o}/\alpha)\Big|^{+}_{\Sigma_1} = \dfrac{d\bar{A}_3}{dr}\Big|^{+}_{\Sigma_1} = -\dfrac{\mu \sin^2\theta}{r^2} \end{array}\right\} \tag{C.8}$$

The second characteristic surface Σ_2 is given by $x = x_o >> 1$. If we now carry out the similar analysis for this case, and use the static exterior solution

$$\bar{A}_o = \bar{A}_1 = \bar{A}_2 = 0 \quad , \quad \bar{A}_3 = \frac{\mu \sin^2\theta}{r} \quad ,$$

found in §3.3, the corresponding jump conditions turn out to be

$$\left.\begin{aligned}
&\underline{\mu=0,1}: \quad (\bar{A}_{o,1} - \bar{A}_{1,o})\Big|_{\Sigma_2}^{+} = 0 \quad , \\
&\underline{\mu=2}: \quad (\bar{A}_{o,2} - \bar{A}_{2,o})(\tfrac{1}{\alpha})\Big|_{\Sigma_2}^{+} + (\bar{A}_{1,2} - \bar{A}_{2,1})\Big|_{\Sigma_2}^{+} = 0 \quad , \\
&\underline{\mu=3}: \quad (\bar{A}_{3,1} + \bar{A}_{3,o}/\alpha)\Big|_{\Sigma_2}^{+} = \frac{d\bar{A}_3}{dr}\Big|_{\Sigma_2}^{+} = -\frac{\mu \sin^2\theta}{r^2} \quad .
\end{aligned}\right\} \qquad (C.9)$$

The jump conditions (C.8) and (C.9) across the characteristic surfaces $y = 0$, $x = x_o >> 1$, guarantee a solution in the time-dependent exterior region of the form

$$\bar{A}_o = \bar{A}_1 = \bar{A}_2 = 0 \quad , \quad \bar{A}_3 = \psi(x,y) \sin^2\theta \quad ,$$

where $\psi \approx \mu/r$ on both characteristic surfaces. This is easily seen physically, because the characteristic initial-value problem of Chapter III must have a unique solution for $F_{\mu\nu}$, and since the assumption, $\bar{A}_o = \bar{A}_1 = \bar{A}_2 = 0$, made in Chapter III, satisfies all required conditions (the jump conditions (C.8), (C.9) as well as the field equations) it must be the solution.

APPENDIX D

Imbedding Relations

We first derive several relations for imbedding an arbitrary hypersurface Σ in an $(n+1)$ - dimensional Riemannian space. Greek indices run from 0 to n ; Latin indices, which distinguish quantities defined on the hypersurface, have the range 1 to n .

Let the equations

$$x^\alpha = x^\alpha(\theta^1, \cdots, \theta^n; V) \quad , \quad V = \text{const.} \tag{D.1}$$

represent an orientable hypersurface Σ with tangent base vectors

$$e_{(i)}{}^\alpha = \partial x^\alpha / \partial \theta^i \quad , \tag{D.2}$$

and unit normal $\underset{\sim}{n}$:

$$\underset{\sim}{n} \cdot \underset{\sim}{e}_{(i)} = 0 \quad , \quad \underset{\sim}{n} \cdot \underset{\sim}{n} = \epsilon(\underset{\sim}{n}) = \begin{cases} +1 & \underset{\sim}{n} \text{ spacelike,} \\ -1 & \underset{\sim}{n} \text{ timelike.} \end{cases} \tag{D.3}$$

As in Chapter II, the extrinsic curvature tensor K_{ab} , and the intrinsic affine connection $\Gamma_{ab}{}^c$ are defined by

$$\left. \begin{aligned} \partial \underset{\sim}{n} / \partial \theta^b &= K_b{}^a \, \underset{\sim}{e}_{(a)} \quad , \\ \underset{\sim}{e}_{(a)} \cdot \partial \underset{\sim}{e}_{(b)} / \partial \theta^c &= \Gamma_{a,bc} \quad , \end{aligned} \right\} \tag{D.4}$$

from which we obtain the Gauss-Weingarten equations

$$\partial \underset{\sim}{e}_{(a)}/\partial\theta^b = -\epsilon(\underset{\sim}{n})\, K_{ab}\, \underset{\sim}{n} + \Gamma_{ab}{}^{c}\, \underset{\sim}{e}_{(c)} \quad . \tag{D.5}$$

(Note that in Chapter II, $\sum$ was a timelike hypersurface, hence $\epsilon(\underset{\sim}{n}) = +1$). Then (D.4) and (D.5), together with the Ricci commutation relations,

$$\left(\left(\frac{\partial}{\partial\theta^b}\frac{\partial}{\partial\theta^a} - \frac{\partial}{\partial\theta^c}\frac{\partial}{\partial\theta^b}\right)\underset{\sim}{e}_{(a)}\right)^{\mu} = R^{\mu}{}_{\alpha\beta\gamma}\, e_{(a)}{}^{\alpha}\, \frac{\partial x^{\beta}}{\partial\theta^b}\frac{\partial x^{\gamma}}{\partial\theta^c} \quad , \tag{D.6}$$

lead to the Gauss-Codazzi equations

$$R_{\alpha\beta\gamma\delta}\, e_{(a)}{}^{\alpha} e_{(b)}{}^{\beta} e_{(c)}{}^{\gamma} e_{(d)}{}^{\delta} = R_{abcd} + \epsilon(\underset{\sim}{n})(K_{bc}K_{ad} - K_{ac}K_{db}) \quad , \tag{D.7}$$

$$R_{\alpha\beta\gamma\delta}\, n^{\alpha} e_{(b)}{}^{\beta} e_{(c)}{}^{\gamma} e_{(d)}{}^{\delta} = K_{bc;d} - K_{bd;c} \quad . \tag{D.8}$$

Given a range of values for V, (D.1) then gives rise to a regular family of hypersurfaces, parametrized so that θ^i are constant along the orthogonal trajectories, i.e. $n^{\alpha}\partial\theta^i/\partial x^{\alpha} = 0$. It then follows that

$$\partial\theta^i/\partial x^{\alpha} = g^{ij} e_{(j)\alpha} = e^{(i)}{}_{\alpha} \quad , \tag{D.9}$$

$$\partial x^{\alpha}(\theta,V)/\partial V = \rho n^{\alpha} \quad , \tag{D.10}$$

where ρ is defined by

$$n_\alpha = \epsilon(\underset{\sim}{n})\rho\partial_\alpha V \quad , \tag{D.11}$$

or equivalently

$$\rho^{-1} = [\epsilon(\underset{\sim}{n})g^{\alpha\beta}(\partial_\alpha V)(\partial_\beta V)]^{\frac{1}{2}} \tag{D.12}$$

where ρ^{-1} vanishes nowhere in the region of interest (see §4.6). It follows from (D.12) that

$$\rho^2 V_{|\alpha\beta} n^\beta = -\partial_\alpha \rho \quad . \tag{D.13}$$

Using (D.4) and (D.10) we find

$$\partial \underset{\sim}{e}_{(i)}/\partial V = \partial(\rho\underset{\sim}{n})/\partial\theta^i = (\partial_i\rho)\underset{\sim}{n} + \rho\, K_i{}^j\, \underset{\sim}{e}_{(j)} \quad , \tag{D.14}$$

from which

$$\partial g_{ab}/\partial V = (\partial/\partial V)(\underset{\sim}{e}_{(a)}\cdot\underset{\sim}{e}_{(b)}) = 2\rho\, K_{ab} \quad , \tag{D.15}$$

$$\partial\underset{\sim}{n}/\partial V = -\,\epsilon(\underset{\sim}{n})(\partial_a\rho)\underset{\sim}{e}^{(a)} . \tag{D.16}$$

From (D.4) we also have

$$K_{ab} = \underset{\sim}{e}_{(a)}\cdot\partial\underset{\sim}{n}/\partial\theta^b = e_{(a)}{}^\alpha n_{\alpha|\beta} e_{(b)}{}^\beta = \rho e_{(a)}{}^\alpha e_{(b)}{}^\beta V_{|\alpha\beta} \quad , \tag{D.17}$$

which, together with (D.13) and the completeness relation

$$g^{\alpha\beta} = g^{ab} e_{(a)}{}^\alpha e_{(b)}{}^\beta + \epsilon(\underset{\sim}{n}) n^\alpha n^\beta \quad , \tag{D.18}$$

yields

$$V_{|\alpha\beta} = \rho^{-1} K_{ab} e^{(a)}{}_{\alpha} e^{(b)}{}_{\beta} - \epsilon(\underset{\sim}{n})\rho^{-2}(\partial_a \rho)(e^{(a)}{}_{\alpha} n_{\beta} + e^{(a)}{}_{\beta} n^{\alpha})$$

$$- \epsilon(\underset{\sim}{n})\rho^{-3}(\partial\rho/\partial V)\, n_{\alpha} n_{\beta} \quad . \tag{D.19}$$

Then for the mean curvature $K = g^{ab} K_{ab}$, we have

$$K = \rho g^{\alpha\beta} V_{|\alpha\beta} + \epsilon(\underset{\sim}{n})\rho^{-2}\ \partial\rho(V,\theta)/\partial V \quad . \tag{D.20}$$

If we now form

$$((\frac{\partial}{\partial V}\frac{\partial}{\partial\theta^{i}} - \frac{\partial}{\partial\theta^{i}}\frac{\partial}{\partial V})\underset{\sim}{n})^{\alpha} \quad ,$$

and use (D.4) , (D.5) , (D.6) , (D.14) and (D.16) , we obtain

$$\rho\ R_{\alpha\mu\nu\beta} e_{(a)}{}^{\alpha} n^{\mu} n^{\nu} e_{(b)}{}^{\beta} = \epsilon(\underset{\sim}{n})\rho_{;ab} + g_{ap}\ \partial K_{b}{}^{p}/\partial V + \rho\ K_{ap} K_{b}{}^{p} \quad . \tag{D.21}$$

By suitably contracting (D.7) , (D.8) and (D.21) , and using (D.18) , we arrive at the decomposition of the Ricci tensor, $R_{\alpha\beta} = R^{\mu}{}_{\alpha\beta\mu}$, and the associated Einstein tensor, $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R$, with respect to the basis $\{\underset{\sim}{e}_{(i)}, \underset{\sim}{n}\}$:

$$2G_{\alpha\beta}\ n^{\alpha} n^{\beta} = -\ \epsilon(\underset{\sim}{n}) g^{ab} R_{ab} + K_{ab} K^{ab} - K^{2} \quad , \tag{D.22}$$

$$R_{\alpha\beta} n^{\alpha} e_{(b)}{}^{\beta} = \partial_{b} K - K_{b}{}^{c}{}_{;c} \quad , \tag{D.23}$$

$$R_{\alpha\beta} e_{(a)}{}^{\alpha} e_{(b)}{}^{\beta} = R_{ab} + \rho^{-1}\rho_{;ab} + \epsilon(\underset{\sim}{n}) K\ K_{ab} + \epsilon(\underset{\sim}{n})\rho^{-1} g_{ap}\ \partial K_{b}{}^{p}/\partial V \ . \tag{D.24}$$

Let us now consider the case where $n = 3$. In order to have agreement with §4.2, we change our notation so that capitalized Latin indices range from 0 to 3, while Greek indices run 1 to 3. For the spacelike hypersurface $t = \text{const.}$ $(\epsilon(\underset{\sim}{n}) = -1)$ it turns out from (D.15) that the extrinsic curvature $\frac{1}{2} V^{-1} \partial g_{\alpha\beta}/\partial t$ vanishes, hence (D.22) - (D.24) become simply

$$\left.\begin{aligned} 2G_{AB}n^A n^B &= g^{\alpha\beta} R_{\alpha\beta} \;, \\ R_{AB} n^A e_{(\beta)}{}^B &= 0 \;, \\ R_{AB} e_{(\alpha)}{}^A e_{(\beta)}{}^B &= R_{\alpha\beta} + V^{-1} V_{|\alpha\beta} \;. \end{aligned}\right\} \tag{D.25}$$

From (D.25) and the Einstein field equations

$$G_{AB} = -\, 8\pi\gamma\, T_{AB} \;,$$

we readily derive the following decomposition:

$$\left.\begin{aligned} \frac{1}{2} g^{\alpha\beta} R_{\alpha\beta} &= 8\pi\gamma\, T^o_o \;, \\ o &= 8\pi\gamma\, T_{\alpha o} \;, \\ G_{\alpha\beta} &= -\, 8\pi\gamma\, T_{\alpha\beta} - V^{-1}(V_{|\alpha\beta} - V^{|\mu}{}_{|\mu} g_{\alpha\beta}) \;. \end{aligned}\right\} \tag{D.26}$$

We are also interested in the 2-space $V = \text{const.}$, considered as imbedded in the 3-space $t = \text{const.}$ This time we use the imbedding relations (D.22) - (D.24), where $n = 2$, $\epsilon(\underset{\sim}{n}) = +1$, and lower case

Latin indices have the range 2,3 .

Let X , Y_a , and Z_{ab} be the right-hand sides of (D.22) , (D.23) , and (D.24) , respectively. Then, by virtue of (4.17) (i.e. from (D.4) , (D.9) , and (D.10)) we have the following identities:

$$\left.\begin{aligned} g^{-1}\partial(gX)/\partial V &= \rho Z_{ab}(K^{ab} - Kg^{ab}) - \rho^{-1}(\rho^2 Y^a)_{;a} \\ g^{-\frac{1}{2}}\,\partial(g^{\frac{1}{2}}Y_a)/\partial V &= [\rho(\delta_a^b\, Z_c^c - Z_a^b)]_{;b} + \rho^{-1}(\rho^2 X)_{;a} \quad , \end{aligned}\right\} \tag{D.27}$$

where g is the determinant of the 2-space metric g_{ab} . Thus, if (4.17) and (D.24) are considered as a system of first order equations for finding g_{ab} and K_{ab} as functions of V , then (D.22) and (D.23) are "involutive constraints", i.e. if they are satisfied on one surface $V =$ const., then from (D.27) they must hold identically.

From (D.19) , (D.24) and (D.26) , we obtain

$$V^{-1}\, g^{-\frac{1}{2}}\,\partial(g^{\frac{1}{2}}\, V\, K_a{}^b)/\partial V = -\rho_{;a}{}^{;b} - \tfrac{1}{2}\,\rho\, R\, \delta_a{}^b - 8\pi\gamma\rho(T_{\alpha\beta}e_{(a)}{}^{\alpha}e_{(b)}{}^{\beta} - \tfrac{1}{2}\, T_A^A\, g_{ab}) \; , \tag{D.28}$$

where $R = g^{ab}\, R_{ab}$, while (D.22) and (D.23) , respectively, combine with (D.19) and (D.26) to give

$$\tfrac{1}{2}(K_{ab}K^{ab} - K^2 - R) = -8\pi\gamma\; T_{\alpha\beta}\; n^{\alpha}n^{\beta} + \rho^{-1}V^{-1}K \quad , \tag{D.29}$$

$$\partial_a K - K_a{}^b{}_{;b} = -8\pi\gamma\; T_{\alpha\beta}\; e_{(a)}{}^{\alpha}n^{\beta} + \rho^{-2}V^{-1}\partial_a\rho \quad . \tag{D.30}$$

Finally, from (D.7), (D.8) and (D.21), with suitable changes in notation, we find

$$\frac{1}{4} R_{ABCD} R^{ABCD} = G_{\alpha\beta} G^{\alpha\beta} + V^{-2} V_{|\alpha\beta} V^{|\alpha\beta} \quad ,$$

which together with (D.19), yields

$$\frac{1}{4} R_{ABCD} R^{ABCD} = G_{\alpha\beta} G^{\alpha\beta} + \rho^{-2} V^{-2} K_{ab} K^{ab} + 2\rho^{-4} V^{-2} \rho_{;a} \rho^{;a}$$

$$+ \rho^{-6} V^{-2} (\partial\rho/\partial V)^2 \ . \tag{D.31}$$

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[14] T. Regge and J.A. Wheeler, Phys. Rev. 108 (1957) 1063. Regge and Wheeler show that all static vacuum perturbations of Schwarzschild's space-time are singular - either at infinity or on the horizon $r = 2m$.

[15] Vanishing of the magnetic dipole moment was treated by V.L. Ginzburg, Dokl. Akad. Nauk SSSR 156 (1964) 43 [transl. as Soviet Phys. Doklady 9 (1964) 329] and by V.L. Ginzburg and L.M. Ozernoi, Zh. Eksperim. i Teor. Fiz. 47 (1964) 1030 [transl. as Soviet Phys. JETP 20 (1965) 689]. Decay of the mass quadrupole moment was also considered by Doroshkevich, Zel'dovich and Novikov (ref. 13).

[16] I.D. Novikov, Zh. Eksperim. i Teor. Fiz. 57 (1969) 949 [transl. as Soviet Phys. JETP 30 (1970) 518].

[17] V. de la Cruz, J.E. Chase and W. Israel, Phys. Rev. Letters 24 (1970) 423. A detailed account of the magnetic dipole part of this work appears in Chapter III, together with a previously unpublished analysis for scalar monopole perturbations.

[18] R. Price, Phys. Rev. (1971), two papers to be published. This work was done independently of ours (ref. 17) but was submitted for publication after ours was in print.

[19] The gravitational dipole is of no consequence, as it can always be made to vanish by choosing the origin of coordinates at the centre of the star.

[20] Most of the material in this section is treated in more detail in NGC, Section III.

[21] In the special case of small deviations from sphericity, this conjecture is true, by Price's Theorem.

[22] The Newtonian collapse of an oblate spheroid of dust to a pancake is analysed by C.C. Lin, L. Mestel, and F.H. Shu, Astrophys. J., 142 (1965) 1431. Thorne points out (NGC, p. 7) that their Newtonian analysis is an excellent approximation to general relativity right down to the pancake endpoint, hence no horizon is formed, and hardly any radiation comes off. The more complicated relativistic collapse of an infinite cylinder of dust to a long thread has recently been studied by T. Morgan and K.S. Thorne, paper in preparation (1971). Again no horizon is formed, and the collapse ends in a relativistic singular thread, with infinite curvature, and a great deal of gravitational radiation.

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[34] Papapetrou and Hamoui assume an equation of state $\sigma - 2P/c^2 = \sigma_o$ (where σ_o and σ are bare-mass density and surface energy density of the shell, respectively). We show in Appendix A, however, that $\sigma - 2P/c^2 < \sigma_o$ for a simple two-dimensional relativistic gas of particles, which only interact by collisions.

[35] K. Kuchar, Czech. Journ. Phys. B18 (1968) 435.

[36] For a more detailed account see, for example, references 30, 32, 35.

[37] Throughout this chapter we assume a system of units in which $G = c = 1$. The signature of the space-time metric is taken to be (+++-) while

that of the hypersurface (++-) . Greek indices run from 1 to 4 ; Latin indices range through 2 , 3 , 4 . A semi-colon denotes covariant differentiation in the timelike hypersurface. $\psi(Q)|^+$ and $\psi(Q)|^-$ denote the limits of the function ψ as the point Q is approached from V_+ and V_- , and the jump discontinuity and mean value of ψ on $\sum$ are given by $[\psi] = \psi|^+ - \psi|^-$ and $\tilde{\psi} = \frac{1}{2}\{\psi|^+ + \psi|^-\}$, respectively.

[38] For more details on surface layers, see, for example W. Israel (ref. 29).

[39] If $\underset{\sim}{A}(u,v)$ is defined on the 2-space $x^\lambda = x^\lambda(u,v)$, then the Ricci commutation relations are given by

$$\left(\left(\frac{\partial}{\partial u}\frac{\partial}{\partial v} - \frac{\partial}{\partial v}\frac{\partial}{\partial u}\right)\underset{\sim}{A}\right)^\alpha = R^\alpha{}_{\beta\lambda\mu}\, A^\beta \frac{\partial x^\lambda}{\partial u}\frac{\partial x^\mu}{\partial v} \quad .$$

See also Appendix D, where several results from this section are derived again for arbitrary (not just timelike) hypersurfaces.

[40] The "$\delta/\delta\tau$" notation in (2.14) is defined, for arbitrary $\underset{\sim}{A}$ by

$$\delta A^\alpha/\delta\tau \equiv (\partial\underset{\sim}{A}/\partial\tau)^\alpha = \partial A^\alpha/\partial\tau + A^\lambda\Gamma^\alpha_{\lambda\mu}\ dx^\mu/d\tau \quad .$$

[41] V. de la Cruz and W. Israel (ref. 32).

[42] K. Kuchar (ref. 35).

[43] B. Hoffman, Quart. Journ. Math. 4 (1933) 179.

[44] With $x^\alpha_+ = (r,\theta,\phi,t_+)$, and the equation of the shell given by $r = R(\tau)$, the matching of the intrinsic and exterior line elements (2.25) and (2.27) leads to $d\tau^2 = f_+(R)dt_+^2 - f_+(R)^{-1}\, dR^2$, from which $dt_+/d\tau = \dot{t}_+ = \{f_+(R) + \dot{R}^2\}^{1/2}/f_+(R)$. Then $u^\alpha_+ = dx^\alpha_+/d\tau = (\dot{R},0,0,\dot{t}_+)$, and since $\underset{\sim}{n}\cdot\underset{\sim}{u} = 0$, we have $\overset{+}{n}_\alpha = (\dot{t}_+,0,0,-\dot{R})$.

[45] V. de la Cruz and W. Israel (ref. 32).

[46] V. de la Cruz and W. Israel (ref. 32).

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[49] Comparing (2.55) with the corresponding Newtonian result, $\Gamma_{class} = M_o^2/16\,\pi R^3$, it becomes clear how the internal energy U contributes to the gravitational self-attraction $(M = M_o + U)$, and also how the non-linearities of the field (R-M in the denominator) strengthen the gravitational forces.

[50] S. Chandrasekhar, Astrophys. J. 140 (1964) 417.

[51] Also inherent in these assumptions are the additional reasonable demands that $m_2 > |e_2|$ and $m_1 > |e_1|$. However, a similar analysis could be carried out for the other possible cases.

[52] K. Kuchar, ref. 35. In Kuchar's case, for equilibrium outside $R = m + (m^2-e^2)^{1/2}$, it is necessary that $P = \Gamma = (M^2-e^2) / 16\,\pi\, R^2(R-M) > 0$, i.e. $M^2 - e^2 > 0$, from which $m^2 - e^2 > 0$ follows.

[53] Throughout this chapter we assume the space-time signature is $(-+++)$. Greek indices will then run from 0 to 3, Latin from 1 to 3.

[54] See for example A.G. Doroshkevich, Ya. B. Zeldovich, and I.D. Novikov (ref. 13), V.L. Ginzburg and L.M. Ozernoi (ref. 15), T. Regge and J.A. Wheeler (ref. 14), W. Israel (refs. 23 and 24).

[55] This is proved in Chapter IV.

[56] This result may now be considered as "almost established" by Carter (ref. 26) and Hawking (ref. 28). For earlier studies in this area see W. Israel, General Relativity and Gravitation, Vol. 2, No. 1 (1971) 53, and K.S. Thorne, Gravitational Collapse - A Review - Tutorial Article (1968, unpublished).

[57] The assumption that R_o is initially very large is purely a matter of convenience. The reasons for this will be made clear in §3.2.

[58] This again is for convenience, since it means that the history of the collapsing shell $(y=0)$ and the shock front $(x=x_o)$ are null hypersurfaces.

[59] A comma here denotes partial differentiation. The coordinates are given by $x^\alpha = (x,y,\theta,\phi)$.

[60] The line element is then simply

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \quad .$$

[61] The general static solution of $\Box\Phi = 0$ on a Schwarzschild background is found in §4.6.

[62] That the similarity of (3.13) and (3.35) is not a coincidence is demonstrated by Price (ref. 18). He shows that, in fact, similar forms occur for weak, zero-mass fields of arbitrary integer spin on a Schwarzschild background.

[63] These numerical integrations were carried out on the University of Alberta IBM 360/67 system, by Mr. R. Teshima. In order to obtain equally-spaced data, we employed the transformation $x = \tan X$, $y = \tan Y$, so that the time-dependent region was given by

$$-\frac{\pi}{2} \leq X \leq \frac{\pi}{2} \quad , \quad 0 \leq Y \leq \frac{\pi}{2} \quad .$$

Over that region, the computer produced a 1000×1000 grid of values for ψ and r .

[64] The free-fall time which is defined to be the approximate time taken by the shell to fall from $r = R_o$ to $r = \lambda m$ (where λ is, say, 3 or 4) is given by

$$t \sim (R_o^{\,3} / Gm)^{1/2} .$$

[65] V.L. Ginzburg, and V.L. Ginzburg and L.M. Ozernoi (ref. 15). The difference there is that Ginzburg considers a "frozen-in" magnetic field which builds up strongly on compression.

[66] In other words, (3.38) can be obtained from the conservation law $((-g)^{1/2} T^{\alpha\beta} \xi_\beta)_{,\alpha} = 0$, where ξ_β is the time-like Killing vector (see 4.2) and $T^{\alpha\beta}$ is the scalar or electromagnetic energy tensor.

[67] We assume that ψ^2 does not blow up on the horizon $(r=1)$ or at $r=\infty$ in a way that overrides the vanishing of f there. Otherwise we would have true singularities on the horizon (contradicting that the singularity there is merely an apparent one) or at infinity (in which case the field would not vanish asymptotically). See also Novikov (ref. 16).

[68] The actual total radiated energy in this case is not $3\mu^2/8$ but $\mu^2/4$ (i.e. the constant factor here is $\frac{2}{3}$). We thus have $I_1 + I_2 = \mu^2/4$ where $I_1 = \frac{2}{3} \int_{-\infty}^{\infty} \psi_x^{\,2} \Big|_{y=\infty} dx$ and $I_2 = \frac{2}{3} \int_0^{\infty} \psi_y^{\,2} \Big|_{x=-\infty} dy$. The numerical results are then $I_1 = .010\ \mu^2$, $I_2 = .240\ \mu^2$. The difference between the magnetostatic energy $\frac{1}{3}\mu^2$ initially present outside $r = 1$, and the total radiated energy $\mu^2/4$, is exactly accounted for as the work done by the difference of the electromagnetic stresses on the two sides of the shell, and goes into increasing the kinetic energy of the collapsing body.

[69] W. Israel (ref. 56).

[70] R. Price (ref. 18).

[71] J.M. Bardeen, unpublished.

[72] A.G.W. Cameron, Annual Reviews of Astronomy and Astrophysics 8 (1970) 179.

[73] K.S. Thorne and W.H. Press, "Ringing of a Black Hole", to be published.

[74] V. de la Cruz, J.E. Chase, and W. Israel (ref. 17).

[75] R. Price (ref. 18).

[76] Price's result for a multipole moment of order $\ell > 0$ has recently been corrected to $\ell nt/t^{2\ell+2}$ by J.M. Bardeen (unpublished).

[77] W. Israel (ref. 23).

[78] W. Israel (ref. 24).

[79] A.I. Janis, E.T. Newman, and J. Winicour (ref. 10).

[80] R. Penney, Phys. Rev. 174 (1968) 1578.

[81] A.I. Janis, D.C. Robinson, and J. Winicour, Phys. Rev. 186 (1969) 1729.

[82] For more detailed or more general treatments see W. Israel (refs. 23,24), or Appendix D.

[83] This assumption is justified in §4.6.

[84] See Appendix D.

[85] W. Israel (ref. 23, Appendix) shows that whenever V is harmonic on Σ, a point P where V has zero gradient must be a point of bifurcation of the equipotential surfaces, i.e. the equipotential surfaces are multi-sheeted in a neighborhood of P.

[86] W. Israel (ref. 23).

[87] R. Penney (ref. 80).

[88] A.I. Janis, E.T. Newman, and J. Winicour (ref. 10).

[89] JNW (ref. 10). In our solution, k, m and V are related to the A, r_o, R and μ of JNW by:

$$A = -k/\sqrt{2} \quad , \quad r_o = 2m \quad , \quad V = \left[\frac{2R - r_o(\mu-1)}{2R + r_o(\mu+1)}\right]^{1/2\,\mu} \quad ,$$

$$\mu = (1 + 4\pi\gamma k^2/m^2)^{1/2} \geq 1 \ .$$

[90] R. Penney (ref. 80).

[91] A.I. Janis, D.C. Robinson, and J. Winicour (ref. 81).